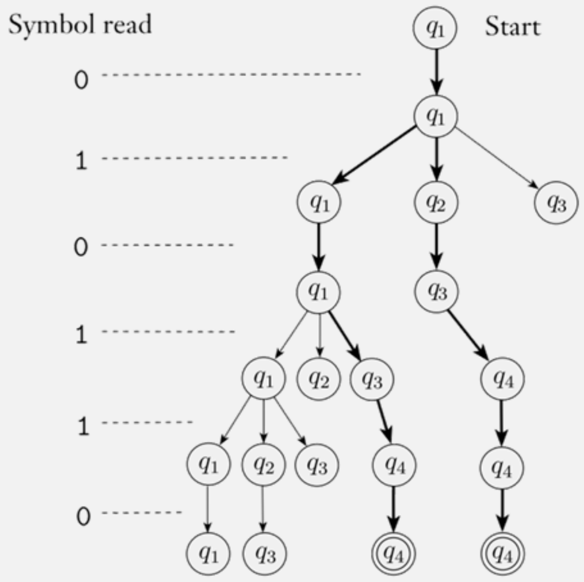
CS 420 / CS 620

Computing with NFAs

Monday, September 29, 2025

UMass Boston Computer Science



Announcements

- HW 3
  - ~~Due: Mon 9/29 12pm (noon)~~
- HW 4
  - Out: Mon 9/29 12pm (noon)
  - Due: Mon 10/6 12pm (noon)

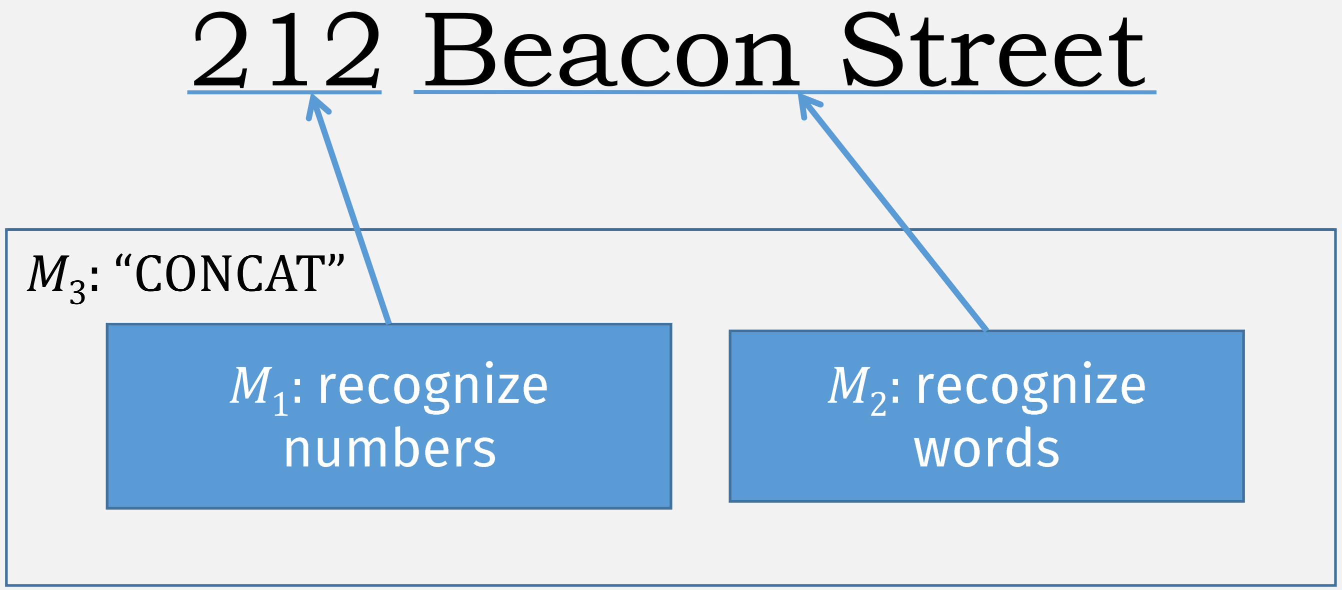
HW 2 Observations

- Don't change the problem
- E.g., Prove the exact theorem given
  - Don't change the wording
  - Don't change the notation
- Note:
  - $L(T) \neq L_T$
  - $L(T)$: all accepted strings of machine T
  - L_T: a given language (set of strings)
- Changed Problem Examples:
  - Proving: "$L(T)$ is a Regular Language"
  - Proving: "L is a Regular Language"
- No outside theorems / notation
  - "The Standard Theorem" ???
  - "The Finite Theorem" ???
- String chars must come from alphabet

Last Time

Another (common string) operation: Concatenation

Example: **Recognizing street addresses**



Last Time

Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Concatenation of Languages

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

If $A = \{\mathtt{fort}, \mathtt{south}\}$ $B = \{\mathtt{point}, \mathtt{boston}\}$

$$A \circ B = \{\mathtt{fortpoint}, \mathtt{fortboston}, \mathtt{southpoint}, \mathtt{southboston}\}$$

Is Concatenation Closed?

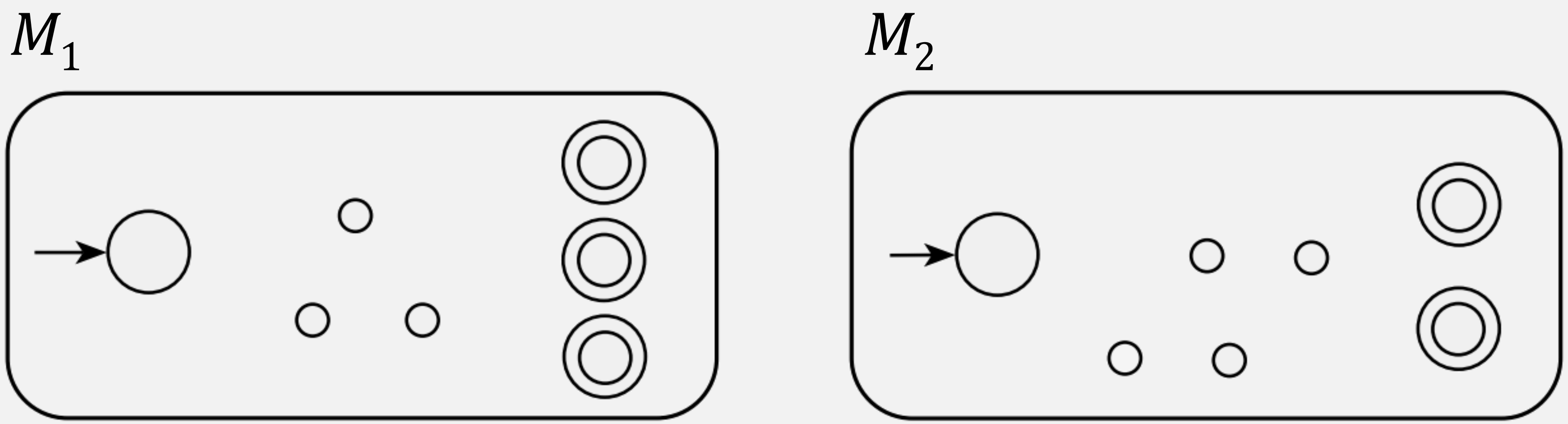
THEOREM

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

- **Construct <u>new</u> machine M recognizing $A_1 \circ A_2$?** (like union)
  - Using **DFA M_1** (which recognizes A_1),
  - and **DFA M_2** (which recognizes A_2)

Concatentation

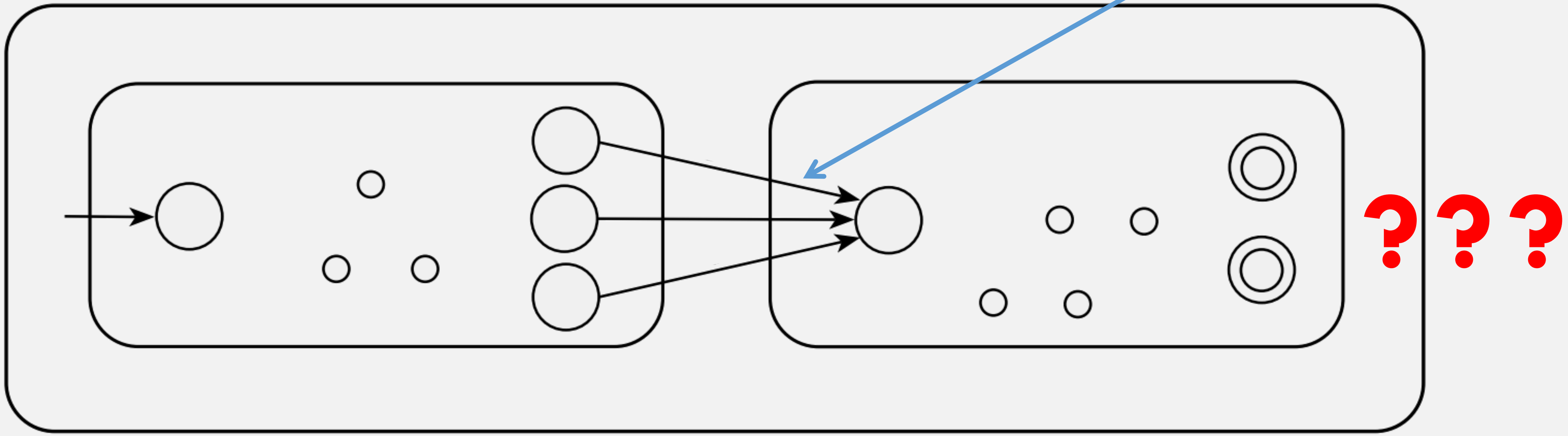


PROBLEM: Can only read input once, can't backtrack

Let M_1 recognize A_1, and M_2 recognize A_2.

Want: Construction of M to recognize $A_1 \circ A_2$

Need to switch machines at some point, but when?



Is Concatenation Closed?

FALSE?

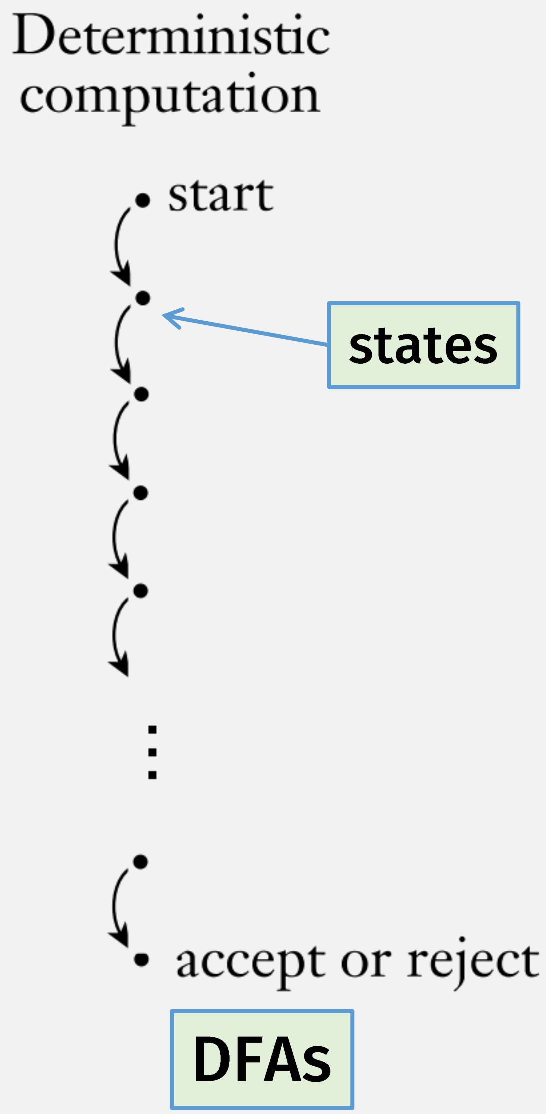
THEOREM

The class of regular languages is closed under the concatenation operation.

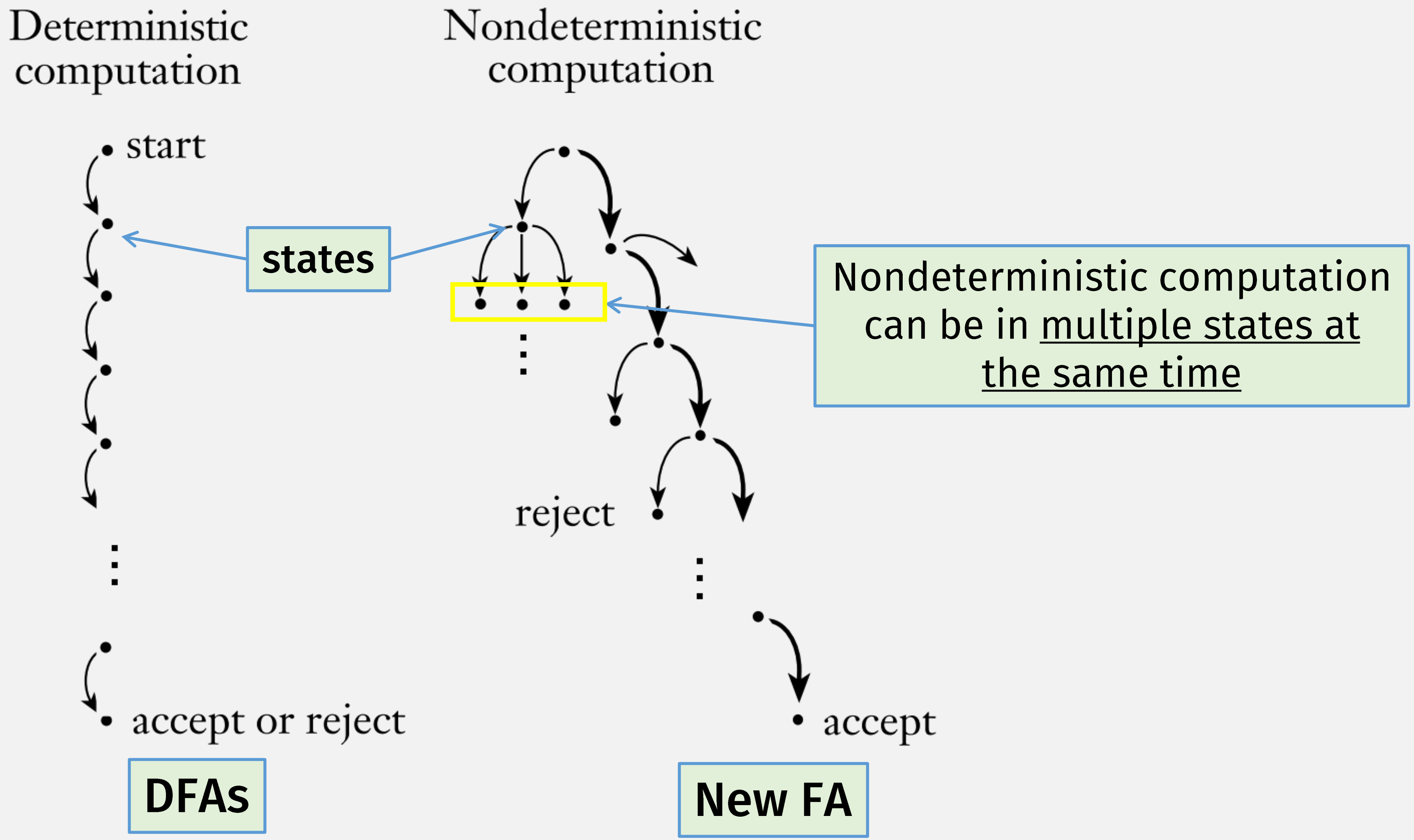
In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

- Cannot **combine A_1 and A_2's machine because:**
  - **Need to switch from A_1 to A_2 at some point …**
  - … but **we don't know when!** (we can only read input once)
- **This requires a new kind of machine!**
- But **does this mean concatenation is not closed for regular langs?**

Deterministic vs Nondeterministic



Deterministic vs Nondeterministic



Previously

DFAs: The Formal Definition

DEFINITION

deterministic

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta\colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

Deterministic Finite Automata (DFA)

Nondeterministic Finite Automata (NFA)

DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Compare with DFA:

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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Difference

Power set, i.e. a transition results in set of states

Power Sets

- A **power set** is the set of all subsets of a set

- Example: S = {a, b, c}

- Power set of S =
  - { { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }
  - Note: includes the empty set!

Nondeterministic Finite Automata (NFA)

DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

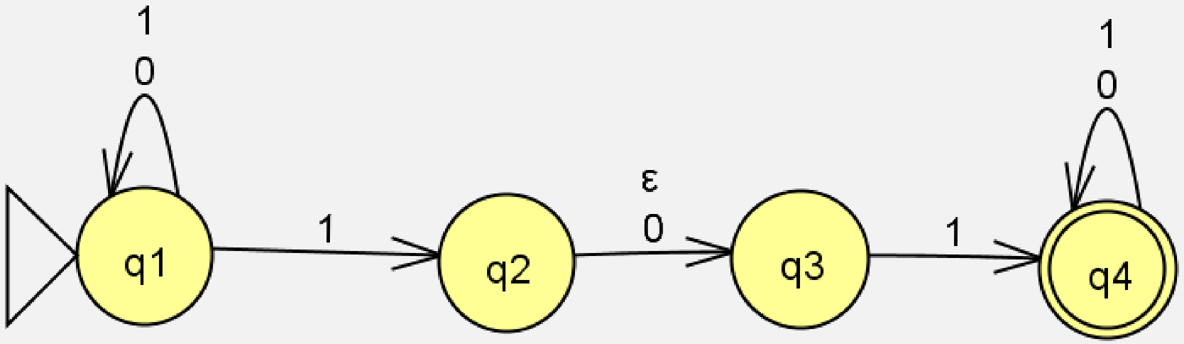
Transition label can be "empty", i.e., machine can transition without reading input

$$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$$

CAREFUL:
ε symbol is reused here, as a transition label (ie, an argument to δ)
- **It's not the empty string!**
- And it's (still) not a character in the alphabet Σ!

NFA Example

- Come up with a formal description of the following NFA:



DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

$$\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$$

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

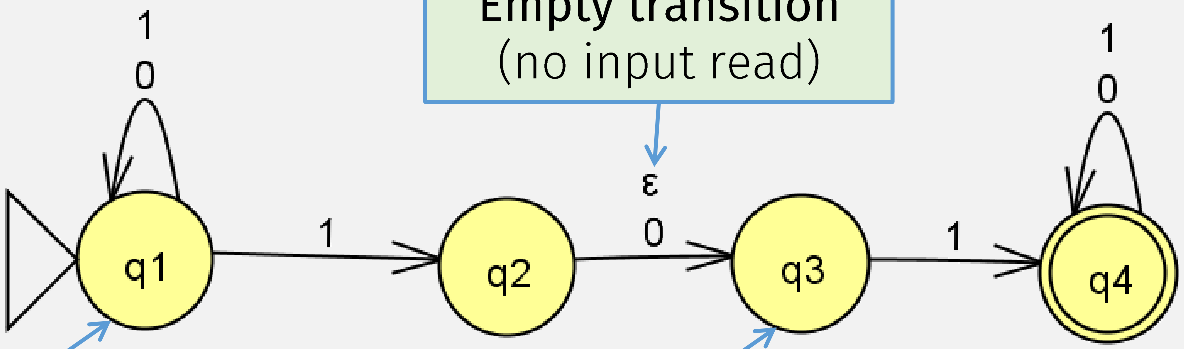
Empty transition
(no input read)

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

Result of transition
is a set

4. q_1 is the start state, and
5. $F = \{q_4\}$.

Empty transition
(no input read)

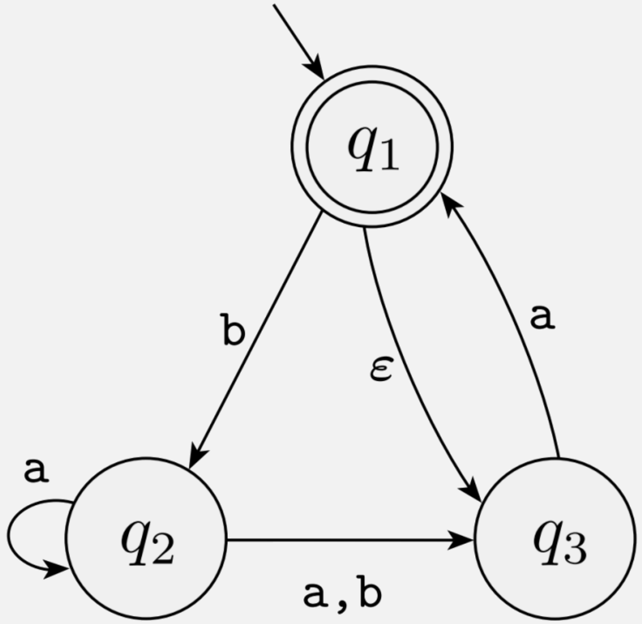


Multiple 1 transitions

No 0 transition

In-class Exercise

- Come up with a formal description for the following NFA
  - $\Sigma = \{ \texttt{a}, \texttt{b} \}$



DEFINITION

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In-class Exercise Solution

Differences with DFA?
- δ output is a set
- state doesn't need transition for every alphabet symbol
- state can have multiple transitions for one symbol
- can have "empty" transitions (δ output is empty set)

Let $N = (Q, \Sigma, \delta, q_0, F)$

- $Q = \{ q_1, q_2, q_3 \}$
- $\Sigma = \{ \mathtt{a}, \mathtt{b} \}$
- δ ... →
- $q_0 = q_1$
- $F = \{ q_1 \}$

$\delta(q_1, \mathtt{a}) = \{ \}$

$\delta(q_1, \mathtt{b}) = \{ q_2 \}$

$\delta(q_1, \varepsilon) = \{ q_3 \}$

$\delta(q_2, \mathtt{a}) = \{ q_2, q_3 \}$

$\delta(q_2, \mathtt{b}) = \{ q_3 \}$

$\delta(q_2, \varepsilon) = \{ \}$

$\delta(q_3, \mathtt{a}) = \{ q_1 \}$

$\delta(q_3, \mathtt{b}) = \{ \}$

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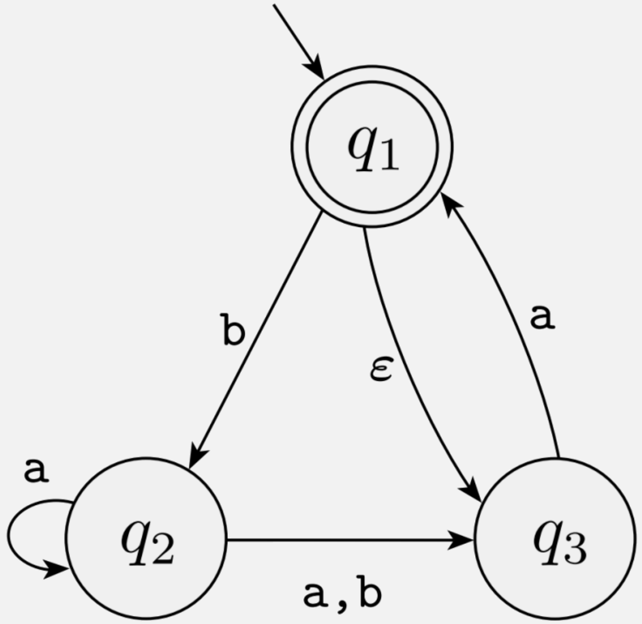
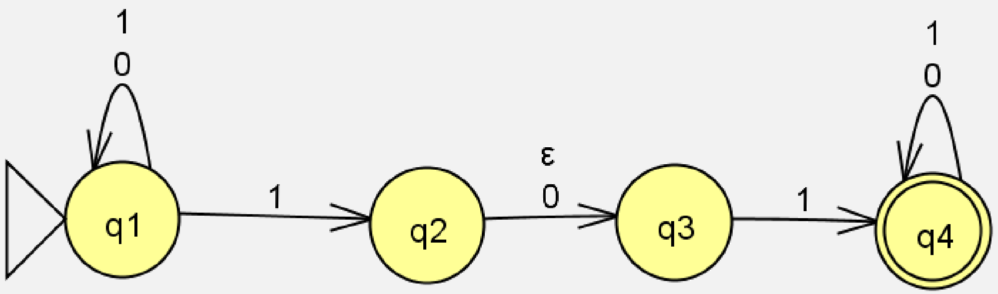
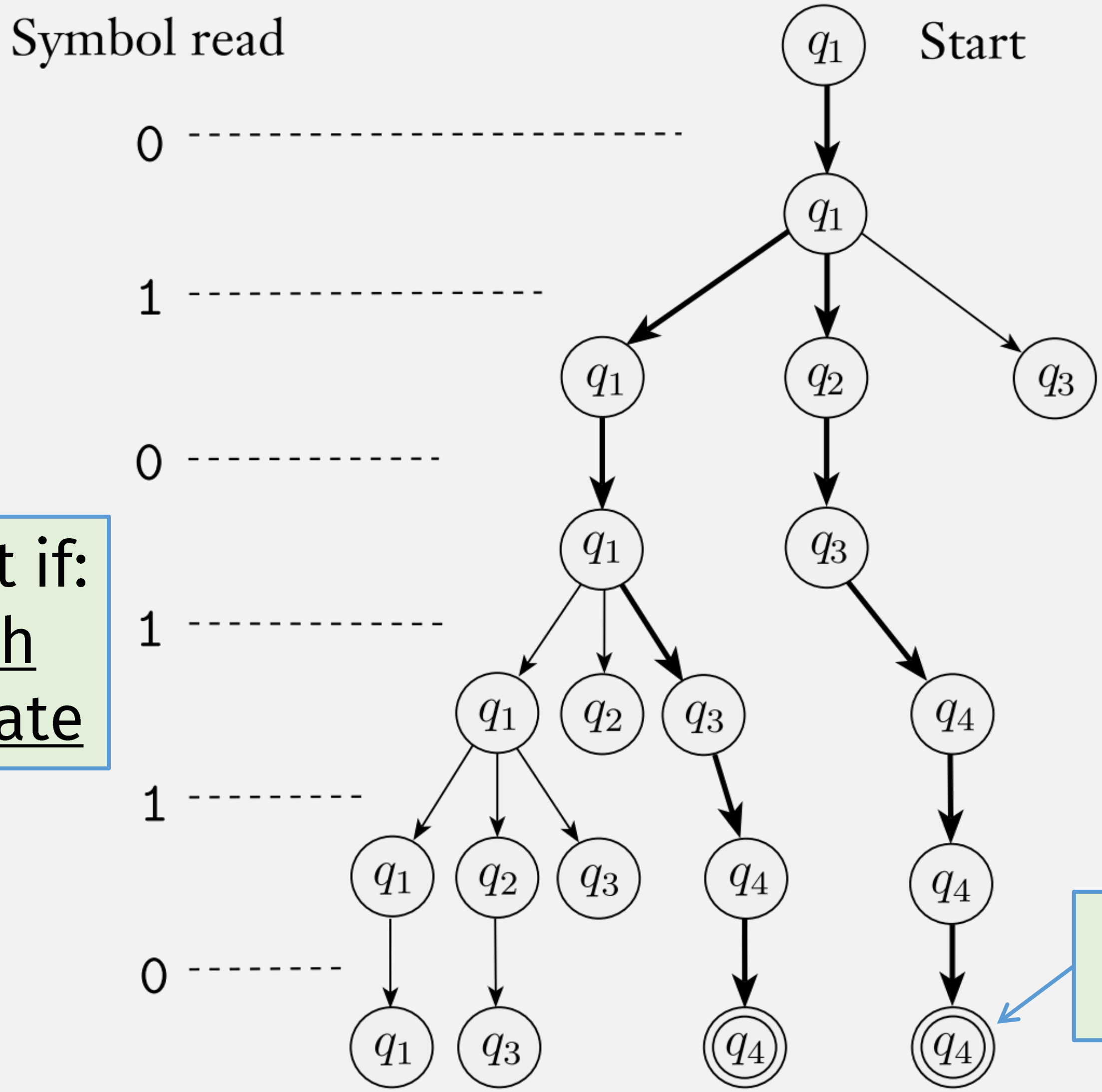


NFA Computation (JFLAP demo): `010110`



NFA Computation Sequence



NFA **accepts** input if: at least <u>one path ends in accept state</u>

Each step can branch into <u>multiple states</u> simultaneously!

This is an **accepting computation**

Flashback

DFA Computation Rules

Informally	*Formally (i.e., mathematically)*
Given	
• A **DFA** (~ a "Program")	• $M = (Q, \Sigma, \delta, q_0, F)$
• and **Input = string of chars**, e.g. "`1101`"	• $w = w_1 w_2 \cdots w_n$
A **DFA computation** (~ "Program run"):	A DFA **computation** is a sequence of states:
• Start in *start state*	

• Repeat:
  • Read **1 char** from **Input,** and
  • Change state according to *transition rules*
	• specified by $\hat{\delta}(q_0, w)$ where:

Result of **computation:**
  • Accept if **last state** is *Accept state*
  • Reject otherwise
	• *M* **accepts** *w* if $\hat{\delta}(q_0, w) \in F$
	• *M* **rejects** otherwise

Flashback

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<u>Result</u> of **computation:**
    • <u>Accept</u> if **last state** is *Accept state*
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	• M **accepts** w if $\hat{\delta}(q_0, w) \in F$
	• M **rejects** otherwise

Ignoring ε transitions, for now!

NFA Computation Rules

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• Start in *start state*	
• Repeat:	• specified by $\hat{\delta}(q_0, w)$ where:
• Read **1 char** from **Input**, and	
For each "current" state, go to next states according to *transition rules*	
... then combine all "next states"	
Result of computation:	
• Accept if last set of states has accept state	• M **accepts** w if ...
• Reject otherwise	• M **rejects** ...

Ignoring ε transitions, for now!

NFA Computation Rules

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- An NFA (~ a "Program")
- and Input = string of chars, e.g. "1101"

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- Start in *start state*
- Repeat:
  - Read 1 char from Input, and
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  - ... then combine all "next states"

Result of computation:
  - Accept if last set of states has accept state
  - Reject otherwise

An **NFA computation** is a sequence of: sets of states

- specified by $\hat{\delta}(q_0, w)$ where:

???

- M **accepts** w if ...
- M **rejects** ...

Flashback

DFA Multi-Step Transition Function

$$\hat{\delta} : Q \times \Sigma^* \rightarrow Q$$

- Domain (inputs):
  - state $q \in Q$ (doesn't have to be start state)
  - String $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range (output):
  - state $q \in Q$ (doesn't have to be an accept state)

Recursive Input Data needs Recursive Function

Base case

A **String** is either:

- the **empty string** (ε), or
- *xa* (non-empty string) where
  - *x* is a **String**
  - *a* is a "**char**" in Σ

Base case $\hat{\delta}(q, \varepsilon) =$

Flashback

DFA Multi-Step Transition Function

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Recursive case

Recursion on String

(Defined recursively)

Base case $\hat{\delta}(q, \varepsilon) = q$

Recursion on String

"smaller" argument

String

char

String

char

Recursive Case $\hat{\delta}(q, w' w_n) = \delta(\hat{\delta}(q, w'), w_n)$

"second to last" state

where $w' = w_1 \cdots w_{n-1}$

Flashback

$\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***

DFA Multi-Step Transition Function

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(Defined recursively)

Base case $\hat{\delta}(q, \varepsilon) = q$

Recursive Case $\hat{\delta}(q, w' w_n) = \delta(\hat{\delta}(q, w'), w_n)$

where $w' = w_1 \cdots w_{n-1}$

Single step from "second to last" state and last char gets to last state

$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function

NFA Multi-Step Transition Function

$\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$

Result is set of states

- Domain (inputs):
  - state $q \in Q$ (doesn't have to be start state)
  - String $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range (output):

states $qs \subseteq Q$

$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function

NFA Multi-Step Transition Function

$\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$

Result is set of states

- Domain (inputs):
  - state $q \in Q$ (doesn't have to be start state)
  - String $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range (output):
  - states $qs \subseteq Q$

Recursively Defined Input
needs
Recursive Function

(Defined recursively)

Base case $\hat{\delta}(q, \varepsilon) = \{q\}$

A **String** is either:

- the **empty string** (ε), or (Base case)
- *xa* (non-empty string) where
  - *x* is a **String**
  - *a* is a "char" in Σ

$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function

NFA Multi-Step Transition Function

$$\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$$

- Domain (inputs):
  - state $q \in Q$ (doesn't have to be start state)
  - String $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range (output):
  - states $qs \subseteq Q$

Recursively Defined Input
needs
Recursive Function

A **String** is either:

- the **empty string** (ε), or
- *xa* (non-empty string) where
  - *x* is a **String**
  - *a* is a "char" in Σ

Recursive case

Recursive part

(Defined recursively)

Base case $\hat{\delta}(q, \varepsilon) = \{q\}$

Recursive Case $\hat{\delta}(q, w' w_n) =$

where $w' = w_1 \cdots w_{n-1}$

Recursion on recursive part

"second to last" set of states

$\hat{\delta}(q, w') = \{q_1, \ldots, q_k\}$

$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function

NFA Multi-Step Transition Function

$$\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$$

- Domain (inputs):
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  - String $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range (output):
  - states $qs \subseteq Q$

Recursively Defined Input
needs
Recursive Function

A **String** is either:

- the **empty string** (ε), or
- *xa* (**non-empty string**) where
  - *x* is a **String**
  - *a* is a "**char**" in Σ

Last char

(Defined recursively)

Base case $\hat{\delta}(q, \varepsilon) = \{q\}$

For each "second to last" state, take **single step** on last char

Recursive Case

$$\hat{\delta}(q, w' w_n) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$$

where $w' = w_1 \cdots w_{n-1}$

$$\hat{\delta}(q, w') = \{q_1, \ldots, q_k\}$$

$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function

NFA Multi-Step Transition Function

$\hat{\delta} : Q \times \Sigma^* \rightarrow$

- Domain (inpu
  - state $q \in$
  - string $w =$
- Range (outpu

states $qs \subseteq$

(Defined recu

Base case

Given

- An **NFA** (~ a "Program")
- and **Input = string of chars**, e.g. "1101"

A **DFA computation** (~ "Program run"):

- **Start** in *start state*
- **Repeat**:
  - **Read 1 char** from **Input**, and
  - **For each "current" state, go to next states** according to *transition rules*

... then combine all sets of "next states"

This ignores ε transitions!

Recursively Defined Input needs

- the **empty string** (ε), or
- *xa* (non-empty string) where
  - *x* is a **String**
  - *a* is a "**char**" in Σ

Recursive Case

$$\hat{\delta}(q, w'w_n) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$$

where $w' = w_1 \cdots w_{n-1}$

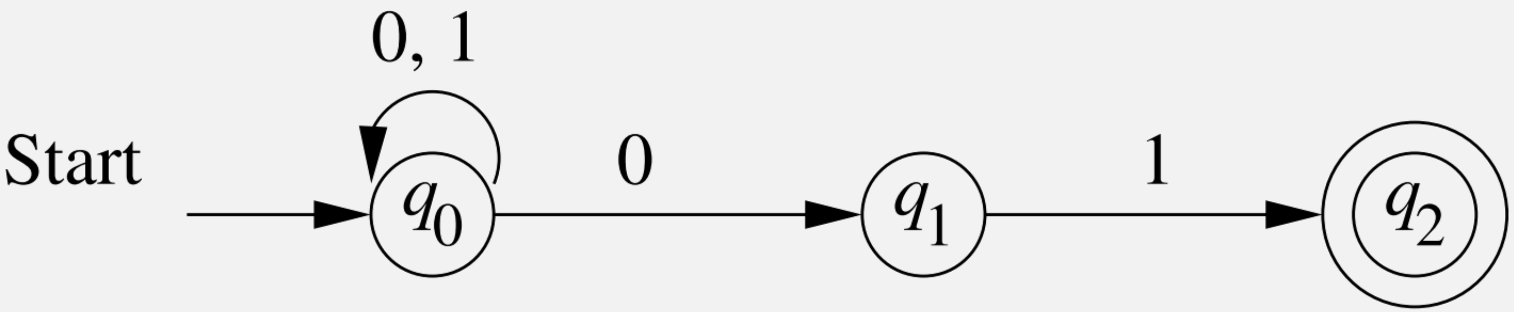
$$\hat{\delta}(q, w') = \{q_1, \ldots, q_k\}$$

NFA Multi-Step δ Example

Base case: $\hat{\delta}(q, \epsilon) = \{q\}$

Recursive case: $\hat{\delta}(q, w) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$

where: $\hat{\delta}(q, w_1 \cdots w_{n-1}) = \{q_1, \ldots, q_k\}$



- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$
- $\hat{\delta}(q_0, 0) = \delta(q_0, 0) = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 00) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 001) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$

We haven't considered **empty transitions!**

Combine result of recursive call with "last step"

Adding Empty Transitions

- Define **the set** $\varepsilon\text{-REACHABLE}(q)$
  - … to be **all states reachable from** q via zero or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

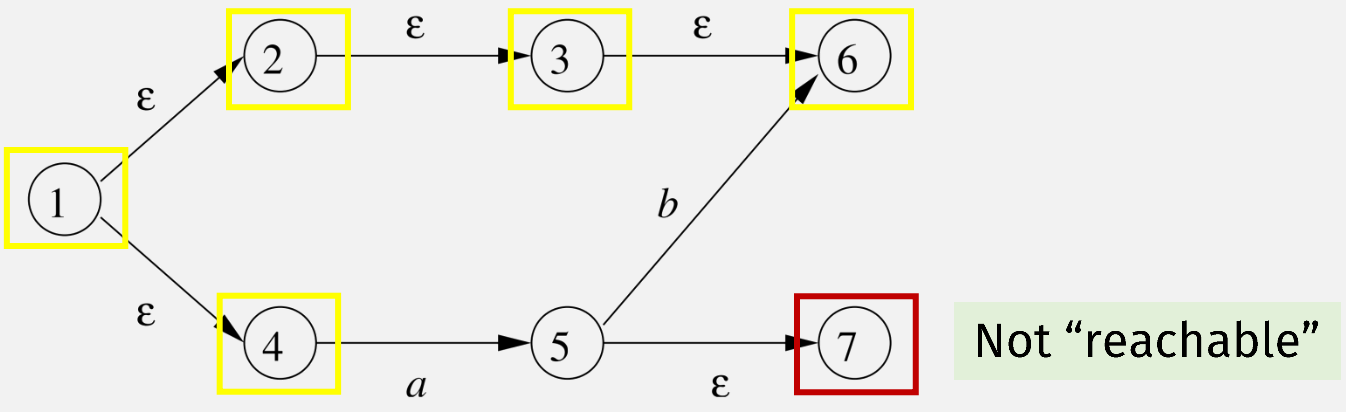
- Recursive case:

A state is in the reachable set if …

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

… there is an empty transition to it from another state in the reachable set

ε-REACHABLE Example



$$\varepsilon\text{-REACHABLE}(1) = \{1, 2, 3, 4, 6\}$$

Handling ε transitions now!

NFA Multi-Step Transition Function

$$\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$$

- Domain (inputs):
  - state $q \in Q$ (doesn't have to be start state)
  - string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range (output):
  - states $qs \subseteq Q$

where $w' = w_1 \cdots w_{n-1}$

$$\hat{\delta}(q, w') = \{q_1, \ldots, q_k\}$$

$$\bigcup_{i=1}^{k} \delta(q_i, w_n) = \{r_1, \ldots, r_\ell\}$$

(Defined recursively)

Base case $\hat{\delta}(q, \varepsilon) = \varepsilon\text{-REACHABLE}(q)$

Recursive Case $\hat{\delta}(q, w' w_n) =$

Handling ε transitions now!

NFA Multi-Step Transition Function

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- Domain (inputs):
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- Range (output):
  - states $qs \subseteq Q$

(Defined recursively)

Base case $\hat{\delta}(q, \varepsilon) = \varepsilon\text{-REACHABLE}(q)$

Recursive Case $\hat{\delta}(q, w' w_n) = \bigcup_{j=1}^{\ell} \varepsilon\text{-REACHABLE}(r_j)$

All chars except last

"second to last" set of states

where $w' = w_1 \cdots w_{n-1}$

$\hat{\delta}(q, w') = \{q_1, \ldots, q_k\}$

$\bigcup_{i=1}^{k} \delta(q_i, w_n) = \{r_1, \ldots, r_\ell\}$

"last" set of states (no ε)

Summary: NFA vs DFA Computation

DFAs

- Can only be in <u>one</u> state
- Transition:
  - <u>Must read 1 char</u>
- Acceptance:
  - If final state <u>is</u> accept state

NFAs

- Can be in <u>multiple</u> states
- Transition
  - <u>Has empty transitions</u>
- Acceptance:
  - If <u>one of</u> final states is accept state

Previously

Concatenation: $A \circ B = \{xy | x \in A \text{ and } y \in B\}$

Is Concatenation Closed?

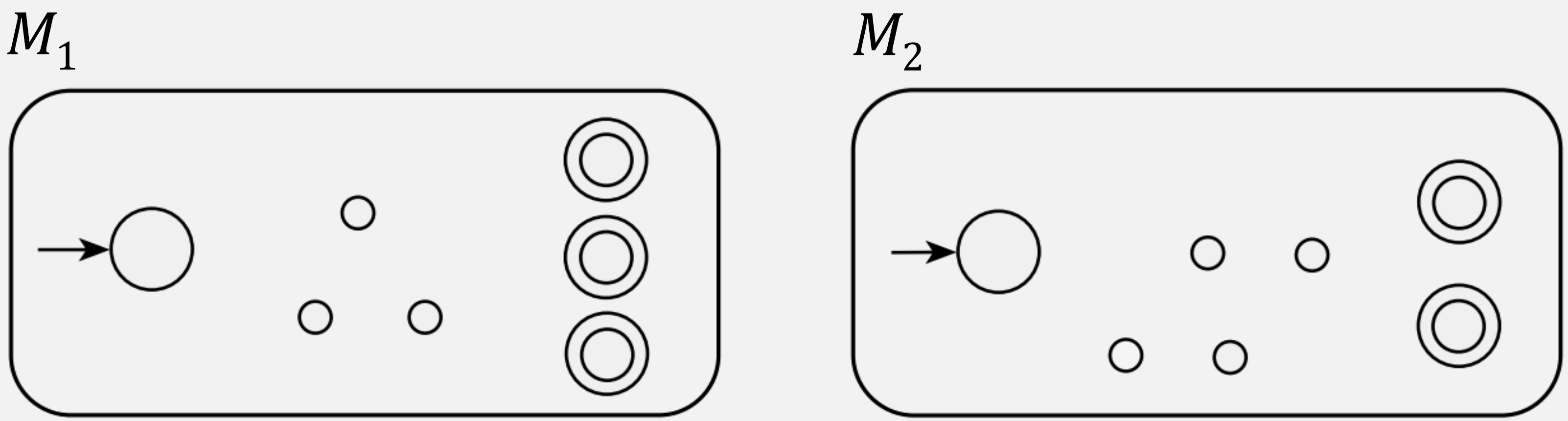
THEOREM

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Proof requires: Constructing new machine

- How does it know when to switch machines?
  - Can only read input once

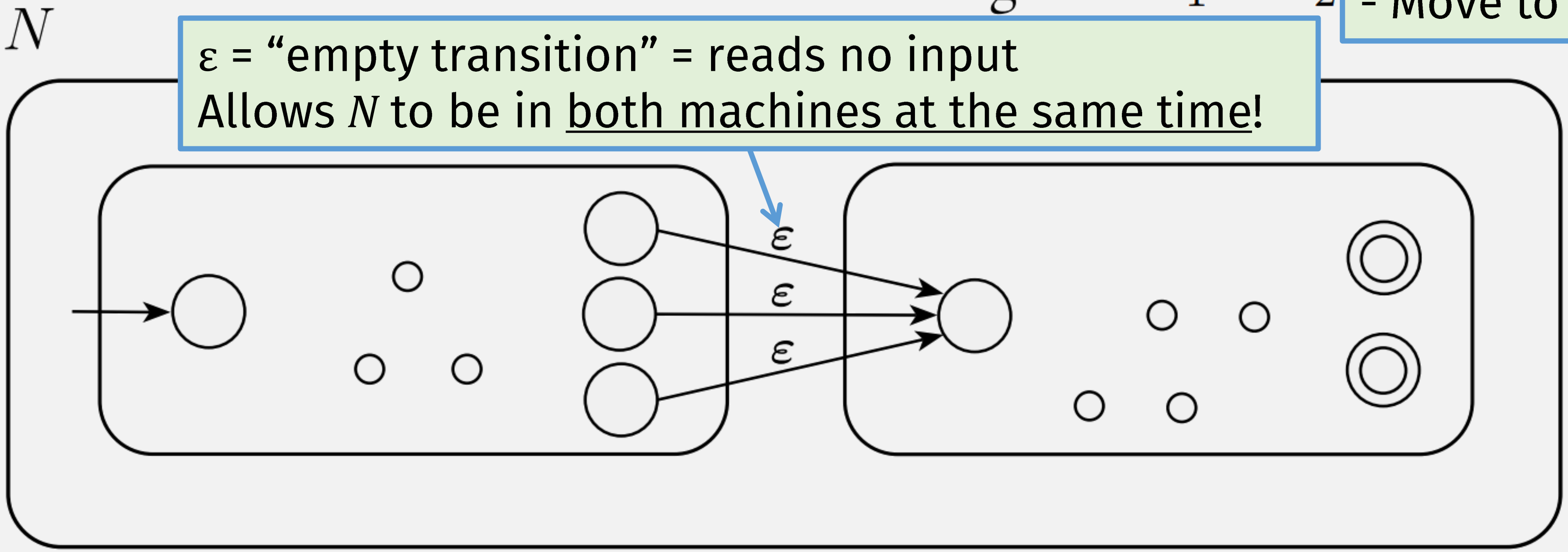


Let M_1 recognize A_1, and M_2 recognize A_2.

<u>Want</u>: Construction of N to recognize $A_1 \circ A_2$

N is an **NFA**! It can:
- Keep checking 1st part with M_1
  and
- Move to M_2 to check 2nd part

ε = "empty transition" = reads no input
Allows N to be in <u>both machines at the same time</u>!



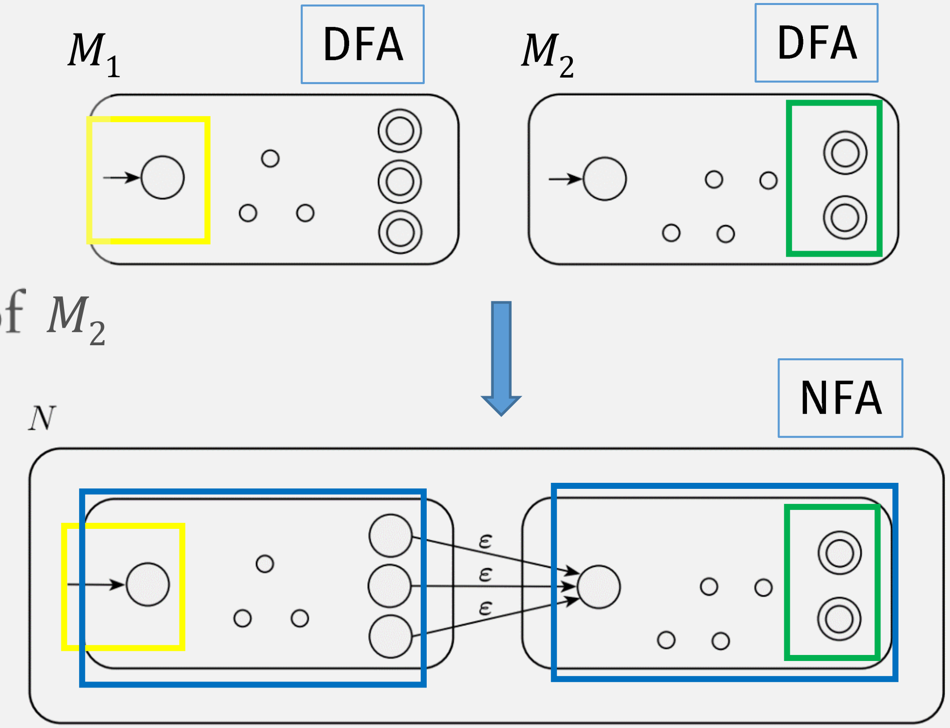
Concatenation is Closed for Regular Langs

PROOF (part of)

Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of M_1
3. The accept states F_2 are the same as the accept states of M_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,



Concatenation is Closed for Regular Langs

Wait, is this true?

PROOF (part of)

Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

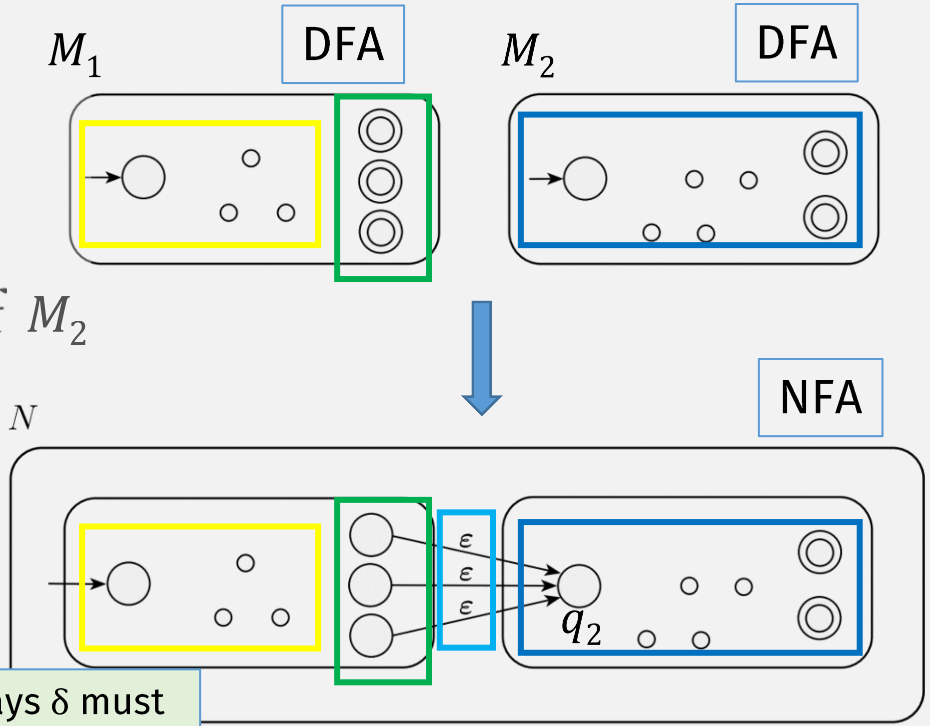
Define the function:

$\text{CONCAT}_{\text{DFA-NFA}}(M_1, M_2) = N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of M_1
3. The accept states F_2 are the same as the accept states of M_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \{\delta_1(q,a)\} \; ? & q \in Q_1 \text{ and } q \notin F_1 \\ \{\delta_1(q,a)\} \; ? & q \in F_1 \text{ and } a \neq \varepsilon \\ ? \quad \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \{\delta_2(q,a)\} \; ? & q \in Q_2. \end{cases}$$

And: $\delta(q, \varepsilon) = \emptyset$, for $q \in Q, q \notin F_1$



NFA def says δ must map every state and ε to set of states

??? ∎

Previously

Is Union Closed For Regular Langs?

Proof

Statements	**Justifications**
1. A_1 and A_2 are regular languages	1. Assumption of If part of If-Then
2. A DFA M_1 recognizes A_1	2. Def of Reg Lang (Coro)
3. A DFA M_2 recognizes A_2	3. Def of Reg Lang (Coro)
4. Construct DFA $M =$ UNION$_{\text{DFA}}$ (M_1,M_2)	4. Def of DFA and UNION$_{\text{DFA}}$
5. M recognizes $A_1 \cup A_2$	5. See Examples Table
6. $A_1 \cup A_2$ is a regular language	6. Def of Regular Language
7. The class of regular languages is closed under the union operation. In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.	7. From stmt #1 and #6

Q.E.D. ■

Is Concat Closed For Regular Langs?

Proof?

Statements	**Justifications**
1. A_1 and A_2 are regular languages	1. Assumption of If part of If-Then
2. A DFA M_1 recognizes A_1	2. Def of Reg Lang (Coro)
3. A DFA M_2 recognizes A_2	3. Def of Reg Lang (Coro)
4. Construct NFA $N = \text{CONCAT}_{\text{DFA-NFA}}(M_1, M_2)$ ☑	4. Def of NFA and $\text{CONCAT}_{\text{DFA-NFA}}$
5. M recognizes ~~$A_1 \cup A_2$~~ $A_1 \circ A_2$	5. See Examples Table
6. ~~$A_1 \cup A_2$~~ $A_1 \circ A_2$ is a regular language	6. ~~Def~~ ??? Does NFA recognize reg langs?
7. The class of regular languages is closed under concatenation operation. In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.	7. From stmt #1 and #6

Q.E.D.?

Previously

If a **DFA recognizes** a language L, then L is a **regular language**

A DFA's Language

- For **DFA** $M = (Q, \Sigma, \delta, q_0, F)$
- M **accepts** w if $\hat{\delta}(q_0, w) \in F$
- M **recognizes** language $\{w \mid M \text{ accepts } w\}$

Definition: A DFA's language is a **regular language**

An NFA's Language?

- For **NFA** $N = (Q, \Sigma, \delta, q_0, F)$

Intersection … … with accept states … … is not empty set

- N ***accepts*** w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$
  - i.e., accept if final states contains <u>at least one</u> accept state

- Language of N = $L(N)$ = $\left\{ w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}$

<u>Q</u>: What kind of languages do NFAs recognize?

Concatenation Closed for Reg Langs?

- **Combining DFAs to recognize concatenation of languages …**

  … **produces** an NFA

- So **to prove concatenation is closed …**

  … **we must prove** that NFAs *also* recognize regular languages.

Specifically, we must prove:

NFAs ⇔ regular languages

"If and only if" Statements

$X \Leftrightarrow Y$ = "X if and only if Y" = X iff Y = X <=> Y

Represents <u>two</u> statements:

1. $\Rightarrow$ if X, then Y
   - "**forward**" direction

2. $\Leftarrow$ if Y, then X
   - "**reverse**" direction

How to Prove an “iff” Statement

$X \Leftrightarrow Y$ = “X if and only if Y” = X iff Y = X <=> Y

Proof has <u>two</u> (If-Then proof) parts:

1. $\Rightarrow$ if X, then Y
   - “**forward**” direction
   - assume X, then use it to prove Y
2. $\Leftarrow$ if Y, then X
   - “**reverse**” direction
   - assume Y, then use it to prove X

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Proof: 2 parts

⇒ If L is regular, then some NFA N recognizes it.

(Easier)

- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 4)

⇐ If an NFA N recognizes L, then L is regular.

Full Statements & Justifications?

"equivalent" = "recognizes the same language"

⇒ If L is regular, then some NFA N recognizes it

Statements	**Justifications**
1. L is a regular language	1. Assumption
2. A DFA M recognizes L	2. Def of Regular lang (Coro)
3. Construct NFA $N = \text{CONVERT}_{\text{DFA-NFA}}(M)$	3. See hw 4!
4. DFA M is **equivalent** to NFA N	4. See Equiv. table!
5. An NFA N recognizes L	5. ???
6. If L is a regular language, then some NFA N recognizes it	6. By Stmts #1 and # 5

Assume the "if" part …

… use it to prove "then" part

"Proving" Machine Equivalence (Table)

Let: DFA $M = (Q, \Sigma, \delta, q_0, F)$
NFA $N = \text{CONVERT}_{\text{DFA-NFA}}(M)$
$\hat{\delta}(q_0, w) \in F$ for some string w

Note:
extra column

String	M accepts?	N accepts?	N accepts? Justification
w	Yes	???	See justification #1
w'	No	???	See justification #2?
…			

If M accepts w …

Then we know …

There is some sequence of states: $r_1 \ldots r_n$, where $r_i \in Q$ and

$$r_1 = q_0 \text{ and } r_n \in F$$

Then N accepts?/rejects? w because …

Justification #1?
There is an accepting sequence of set of states in N … for string w

Exercise left for HW
Show that you know how an NFA computes

"Proving" Machine Equivalence (Table)

Let: DFA $M = (Q, \Sigma, \delta, q_0, F)$

NFA $N = \text{CONVERT}_{\text{DFA-NFA}}(M)$

$\hat{\delta}(q_0, w) \in F$ for some string w

$\hat{\delta}(q_0, w') \notin F$ for some string w'

String	M accepts?	N accepts?	N accepts? Justification
w	Yes	???	See justification #1
w'	No	???	See justification #2?
…			

If M rejects w' …

Then we know …

Exercise left for HW

Show that you know how an NFA computes

Then N accepts?/rejects? w' because …

Justification #2?

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Proof:

☑ ⇒ If L is regular, then some NFA N recognizes it.

(Easier)

- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 4)

⇐ If an NFA N recognizes L, then L is regular.

(Harder)

- We know: for L to be **regular**, there must be a **DFA** recognizing it
- Proof Idea for this part: Convert given NFA N → an equivalent DFA

"equivalent" =
"recognizes the same language"

How to convert **NFA→DFA**?

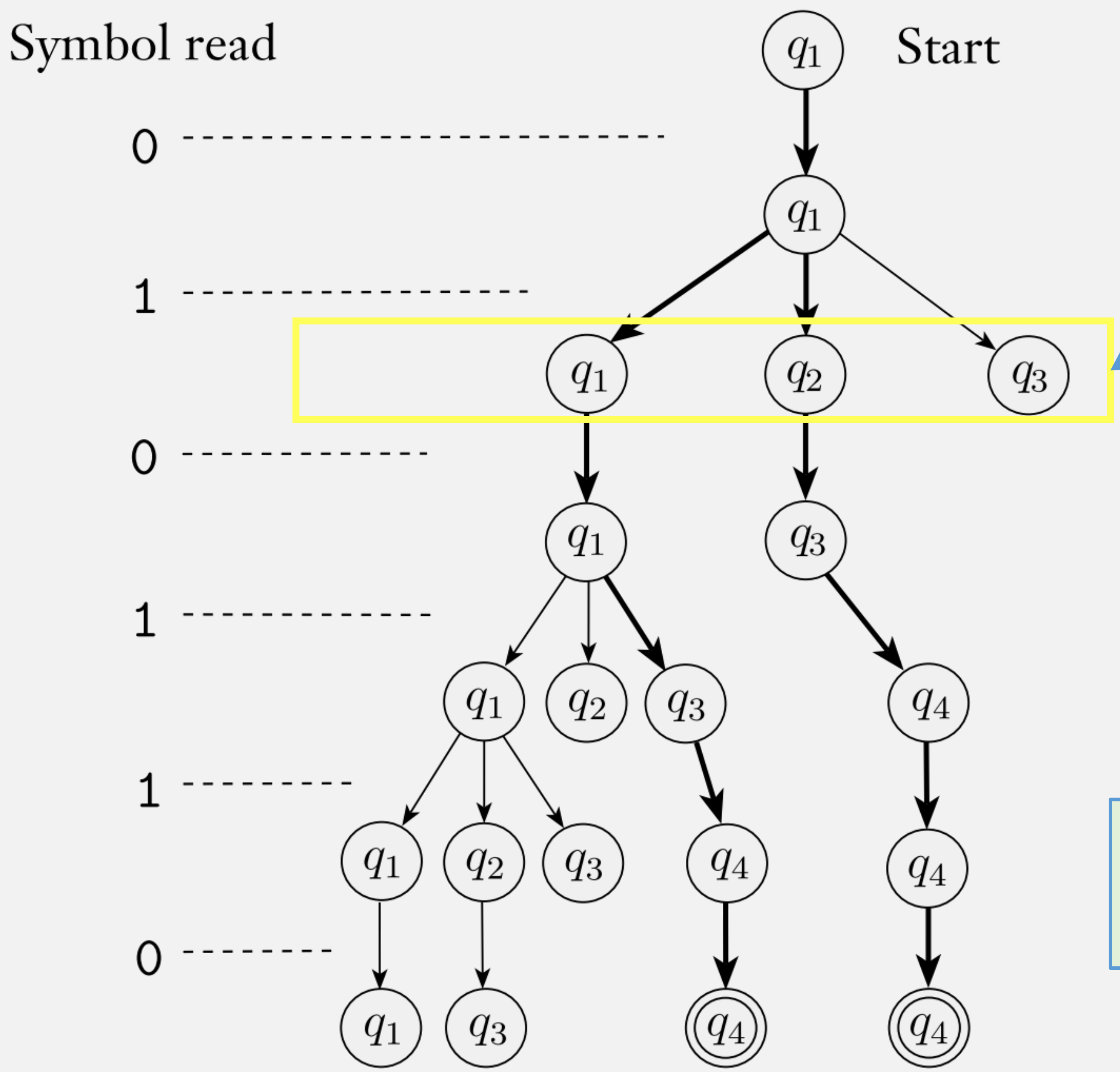
A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Proof idea:
Let each "state" of the DFA
= set of states in the NFA



NFA computation can be in multiple states

DFA computation can only be in one state

So **encode**:
a set of NFA states
as one DFA state

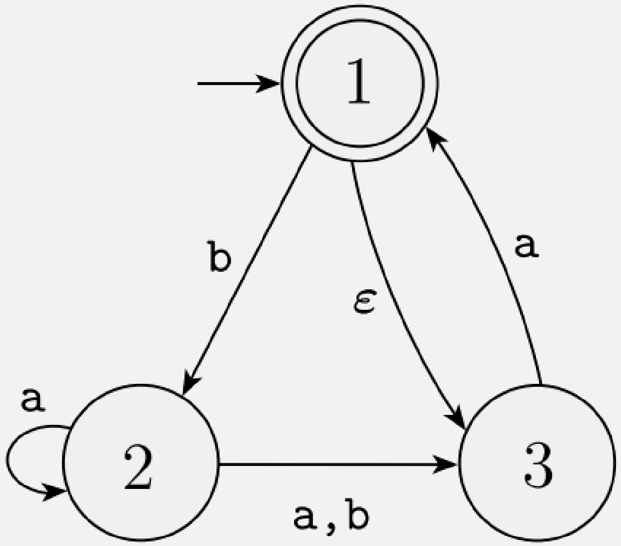
This is similar to the proof strategy from **"Closure of union"** where:
a state = a pair of states

Convert **NFA→DFA**, Formally

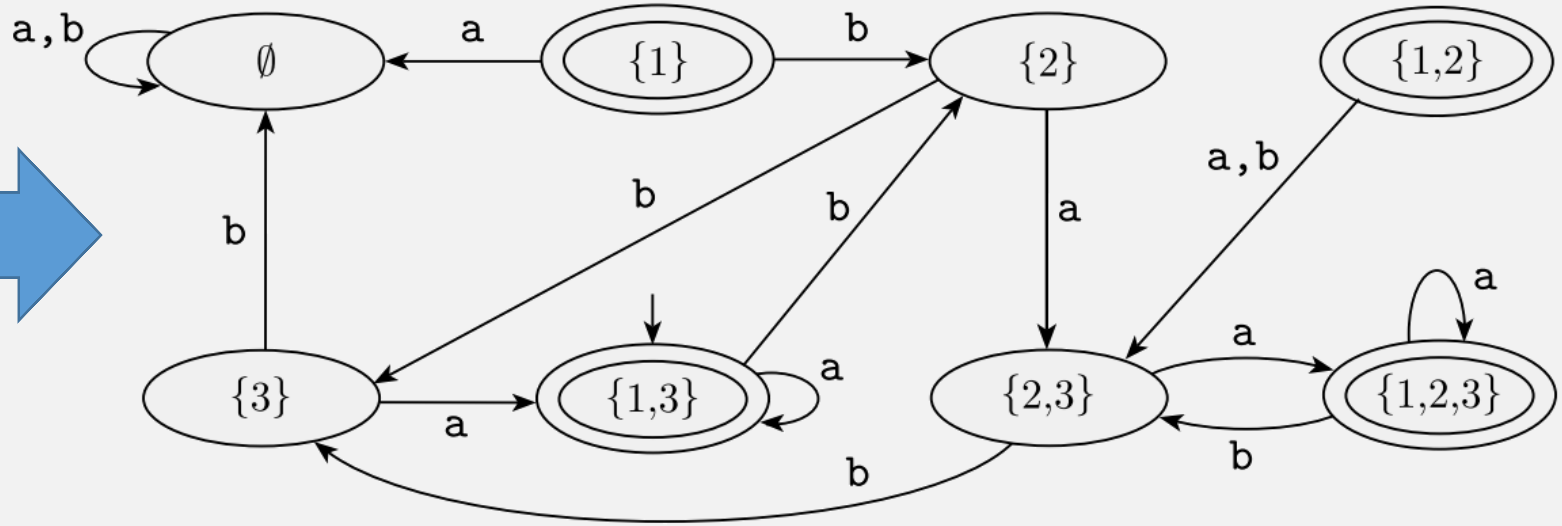
- Let **NFA** N = $(Q, \Sigma, \delta, q_0, F)$
- **An equivalent DFA** M **has states** Q' = $\mathcal{P}(Q)$ (power set of Q)

Example:

- Let **NFA** $N_4 = (Q, \Sigma, \delta, q_0, F)$
- An equivalent DFA D has states $= \mathcal{P}(Q)$ (power set of Q)



The NFA N_4



A DFA D that is equivalent to the NFA N_4

No empty transitions

NFA→DFA

Have: NFA $N = (Q_{NFA}, \Sigma, \delta_{NFA}, q_{0NFA}, F_{NFA})$

Want: DFA $D = (Q_{DFA}, \Sigma, \delta_{DFA}, q_{0DFA}, F_{DFA})$

1. $Q_{DFA} = \mathcal{P}(Q_{NFA})$

A DFA state = a set of NFA states

qs = DFA state = set of NFA states

2. For $qs \in Q_{DFA}$ and $a \in \Sigma$
   - $\delta_{DFA}(qs, a) = \bigcup_{q \in qs} \delta_{NFA}(q, a)$

A DFA step = an NFA step for all states in the set

3. $q_{0DFA} = \{q_{0NFA}\}$

4. $F_{DFA} = \{ qs \in Q_{DFA} \mid qs \text{ contains accept state of } N \}$

Flashback: Adding Empty Transitions

- Define the set $\varepsilon\text{-REACHABLE}(q)$
  - … to be all states reachable from q via zero or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

- Recursive case:

A state is in the reachable set if …

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

… there is an empty transition to it from another state in the reachable set

With empty transitions

NFA→DFA

Have: NFA $N = (Q_{NFA}, \Sigma, \delta_{NFA}, q_{0NFA}, F_{NFA})$

Want: DFA $D = (Q_{DFA}, \Sigma, \delta_{DFA}, q_{0DFA}, F_{DFA})$

Almost the same, except …

1. $Q_{DFA} = \mathcal{P}(Q_{NFA})$
2. For $qs \subseteq Q_{DFA}$ and $a \in \Sigma$
   - $\delta_{DFA}(qs, a) =$ ~~$\bigcup_{q \in qs} \delta_{NFA}(q, a)$~~ $S = \bigcup_{q \in qs} \delta_{NFA}(q, a)$

     $$\bigcup_{s \in S} \varepsilon\text{-REACHABLE}(s)$$
3. $q_{0DFA} =$ ~~$\{q_{0NFA}\}$~~ $\varepsilon\text{-REACHABLE}(q_{0NFA})$
4. $F_{DFA} = \{ qs \in Q_{DFA} \mid qs$ contains accept state of $N \}$

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Statements & Justifications?

Proof:

⇒ If L is regular, then some NFA N recognizes it.

(Easier)

- We know: if L is **regular**, then a **DFA** exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 4)

⇐ If an NFA N recognizes L, then L is regular.

Examples table?

(Harder)

- We know: for L to be **regular**, there must be a **DFA** recognizing it
- Proof Idea for this part: Convert given NFA N → an equivalent DFA …

… using our NFA to DFA algorithm!

■

Concatenation is Closed for Regular Langs ☑

Wait, is this true?

PROOF

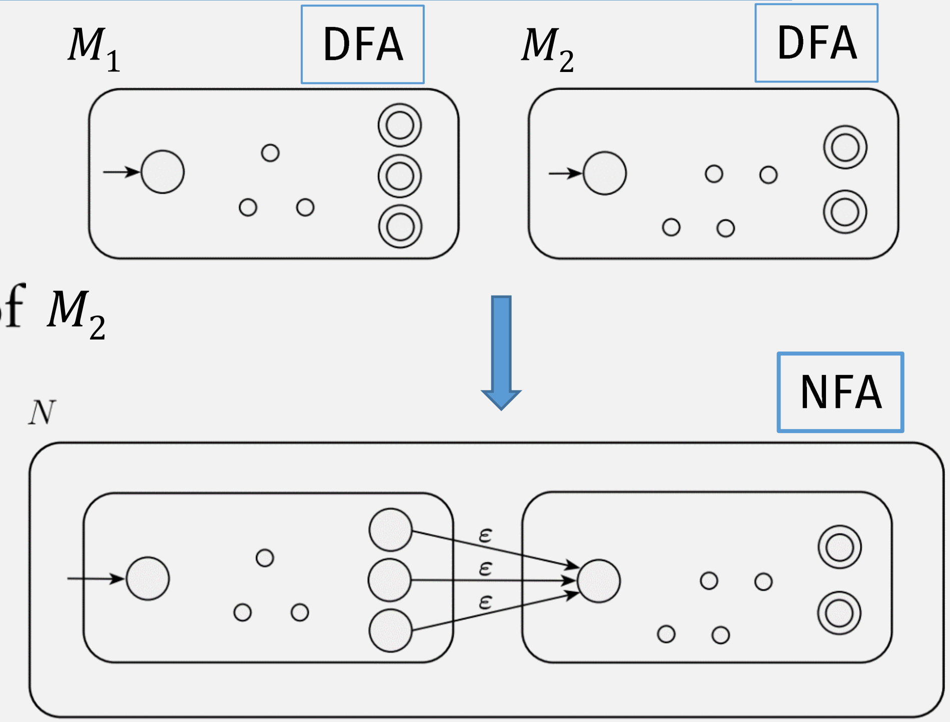
If a language has an **NFA** recognizing it, then **it is a regular** language

Let DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1
DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2

$\text{CONCAT}_{\text{DFA-NFA}}(M_1, M_2) = N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of M_1
3. The accept states F_2 are the same as the accept states of M_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \{\delta_1(q,a)\} & q \in Q_1 \text{ and } q \notin F_1 \\ \{\delta_1(q,a)\} & q \in F_1 \text{ and } a \neq \varepsilon \\ \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \{\delta_2(q,a)\} & q \in Q_2. \end{cases}$$



And: $\delta(q, \varepsilon) = \emptyset$, for $q \in Q, q \notin F_1$ ~~???~~ ■ ☑

New possible proof strategy!

Sipser Theorem 1.47

Concat Closed for Reg Langs: Use NFAs Only

PROOF

If language is **regular**, then it has an **NFA** recognizing it …

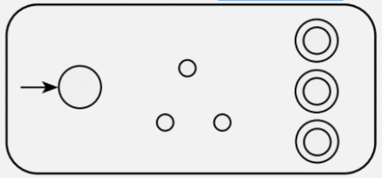
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
NFAs $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

CONCAT$_{NFA}$ $(N_1, N_2) = N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

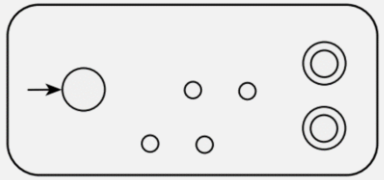
1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of N_1
3. The accept states F_2 are the same as the accept states of N_2 ????
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) \ ? & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) \ ? & q \in F_1 \text{ and } a \neq \varepsilon \\ ? \quad \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) \ ? & q \in Q_2. \end{cases}$$

N_1 NFA

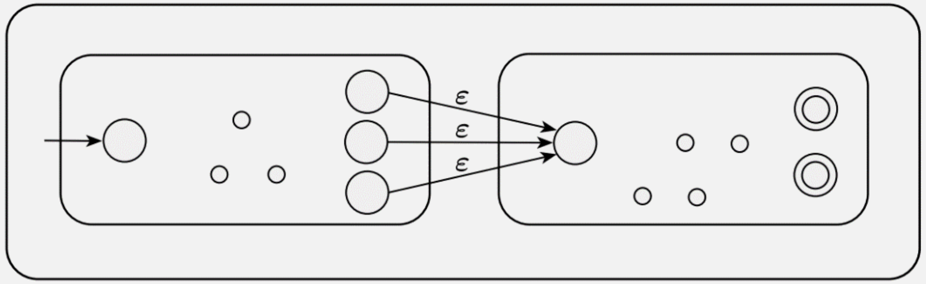


N_2 NFA



All NFAs

N NFA



Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Flashback: Union is Closed For Regular Langs

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Proof:

- How do we prove that a language is regular?
  - Create a DFA or NFA recognizing it!
- Combine the machines recognizing A_1 and A_2
  - Should we create a DFA or NFA?

Flashback: Union is Closed For Regular Langs

Proof

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: $\text{UNION}_{\text{DFA}}$ (M_1, M_2) = M = $(Q, \Sigma, \delta, q_0, F)$ using M_1 and M_2
- states of M: $Q = \{(r_1, r_2) | \, r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2

State in M = M_1 state + M_2 state

- M transition fn: $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$

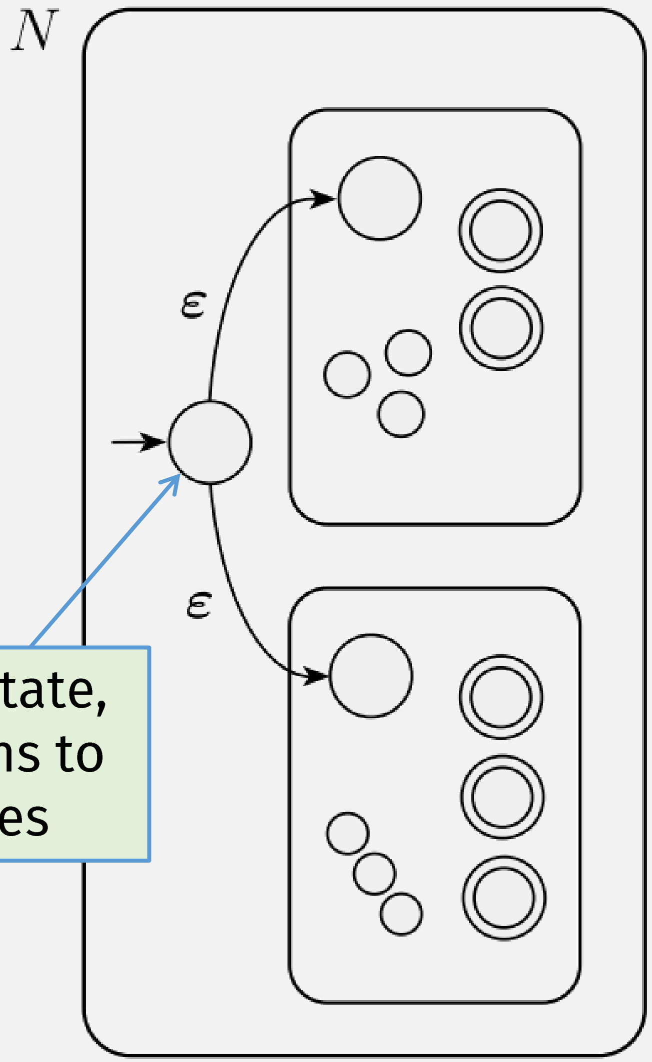
M step = a step in M_1 + a step in M_2

- M start state: (q_1, q_2)

Accept if either M_1 or M_2 accept

- M accept states: $F = \{(r_1, r_2) | \, r_1 \in F_1 \text{ or } r_2 \in F_2\}$

Union is Closed for Regular Languages



Add new start state, and ε-transitions to old start states

Alternate Proof, with NFAs

Union is Closed for Regular Languages

PROOF

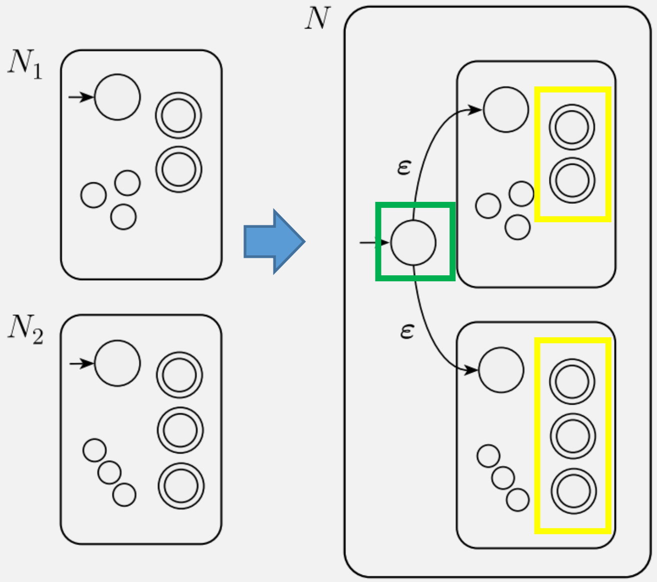
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

UNION$_{NFA}$ $(N_1, N_2) = N =$ $(Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.

2. The state q_0 is the start state of N.

3. The set of accept states $F = F_1 \cup F_2$.



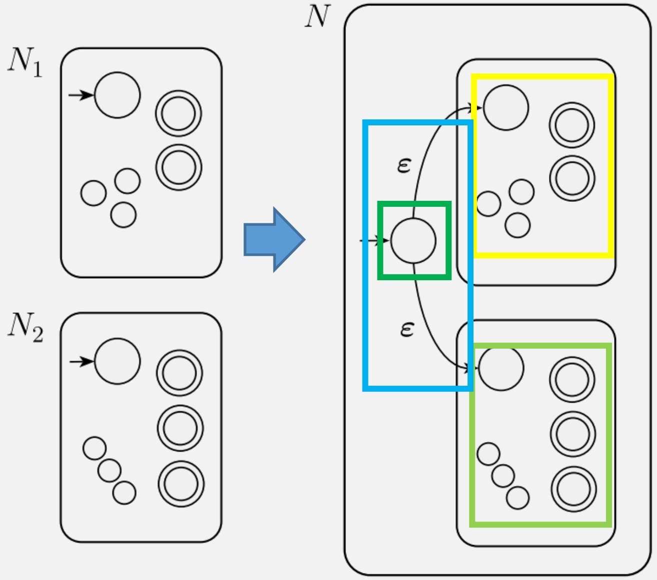
Alternate Proof, with NFAs

Union is Closed for Regular Languages

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

UNION$_{NFA}$ $(N_1, N_2) = N =$ $(Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.



1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) \; ? & q \in Q_1 \\ \delta_2(q,a) \; ? & q \in Q_2 \\ \{q_1, q_2\} \; ? & q = q_0 \text{ and } a = \varepsilon \\ \emptyset \; ? & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

Don't forget Statements and Justifications!

New possible proof strategy!

Sipser Theorem 1.47

Concat Closed for Reg Langs: Use NFAs Only

PROOF

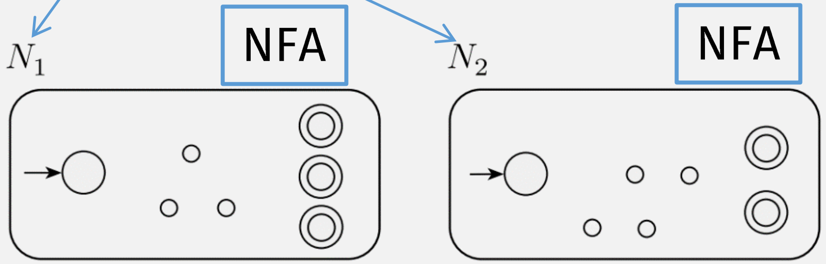
If language is **regular**, then it has an **NFA** recognizing it …

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
NFAs $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

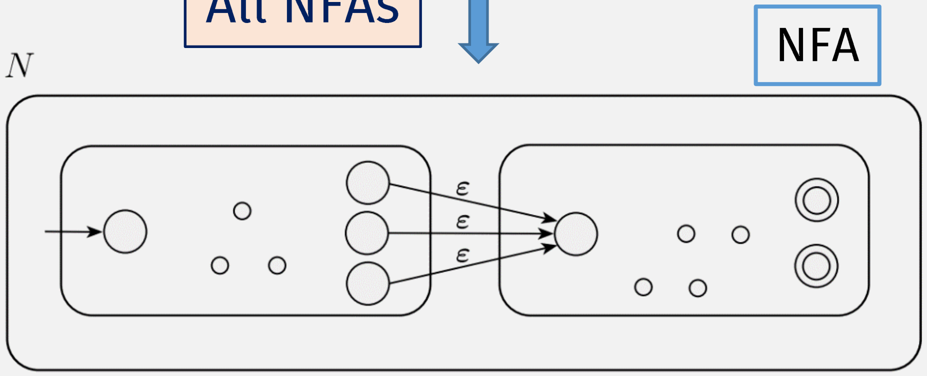
CONCAT$_{NFA}$ $(N_1, N_2) = N =$ $(Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of N_1
3. The accept states F_2 are the same as the accept states of N_2 ????
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) \; ? & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) \; ? & q \in F_1 \text{ and } a \neq \varepsilon \\ ? \quad \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) \; ? & q \in Q_2. \end{cases}$$



All NFAs



List of Closed Ops for Reg Langs (so far)

- ☑ Union
- ☑ Concatentation
- Kleene Star (repetition) ?

Star: $A^* = \{x_1 x_2 \ldots x_k |\ k \geq 0 \text{ and each } x_i \in A\}$

Kleene Star Example

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

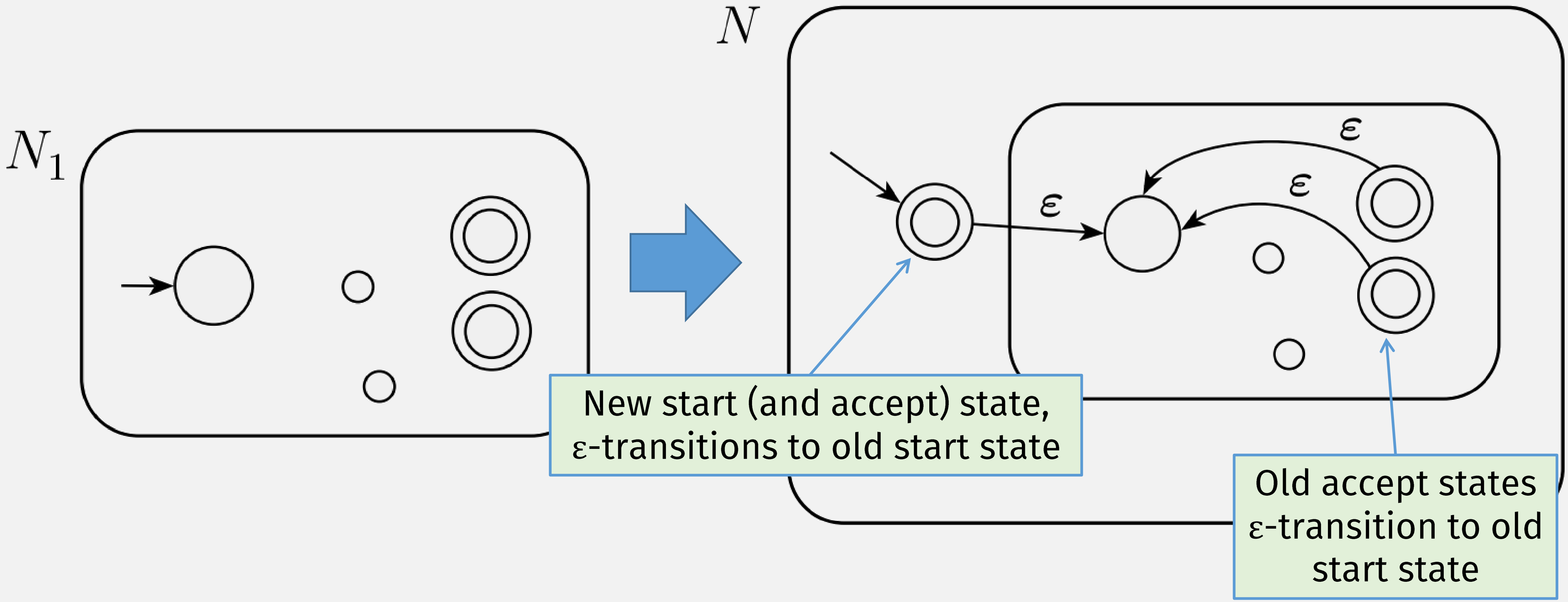
If $A = \{\mathtt{good}, \mathtt{bad}\}$

$$A^* = \{\varepsilon, \text{good, bad, goodgood, goodbad, badgood, badbad, goodgoodgood, goodgoodbad, goodbadgood, goodbadbad}, \ldots\}$$

Note: repeat zero or more times

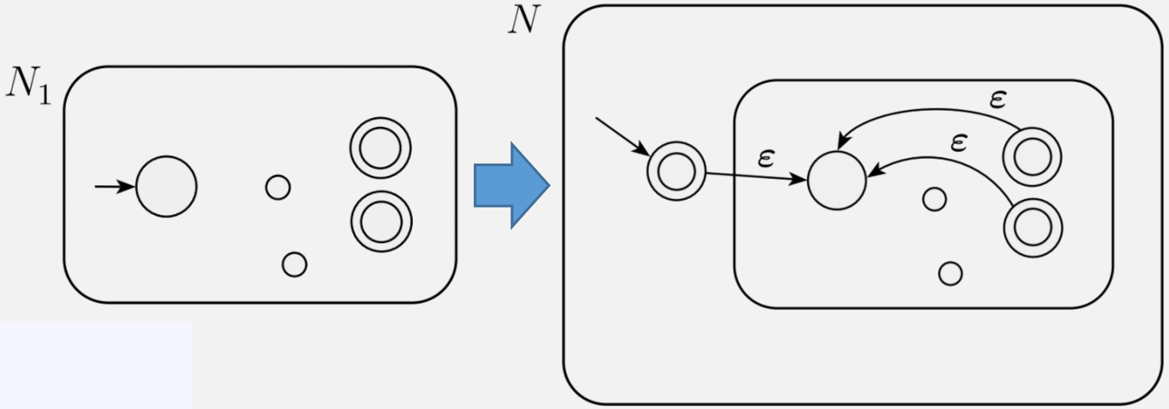
(this is an infinite language!)

Kleene Star


N_1
N
ε
ε
ε
New start (and accept) state,
ε-transitions to old start state
Old accept states
ε-transition to old
start state

In-class exercise

Kleene Star is Closed for Regular Langs



THEOREM

The class of regular languages is closed under the star operation.

In-class solution

Kleene Star is Closed for Regular Langs

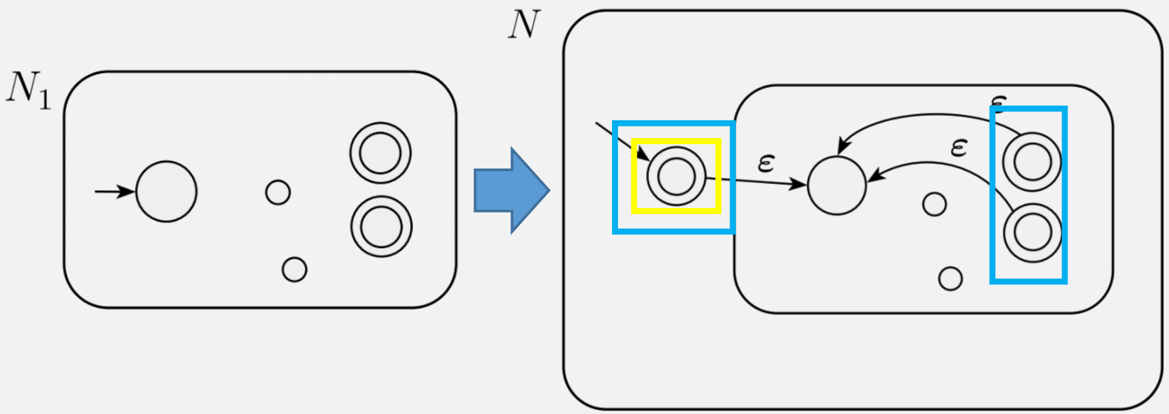
(part of)

PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1.

$N = \text{STAR}_{\text{NFA}}(N_1)$ $= (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.

1. $Q = \{q_0\} \cup Q_1$
2. The state q_0 is the new start state.
3. $F = \{q_0\} \cup F_1$

Kleene star of a language must accept the empty string!



In-class solution

Kleene Star is Closed for Regular Langs

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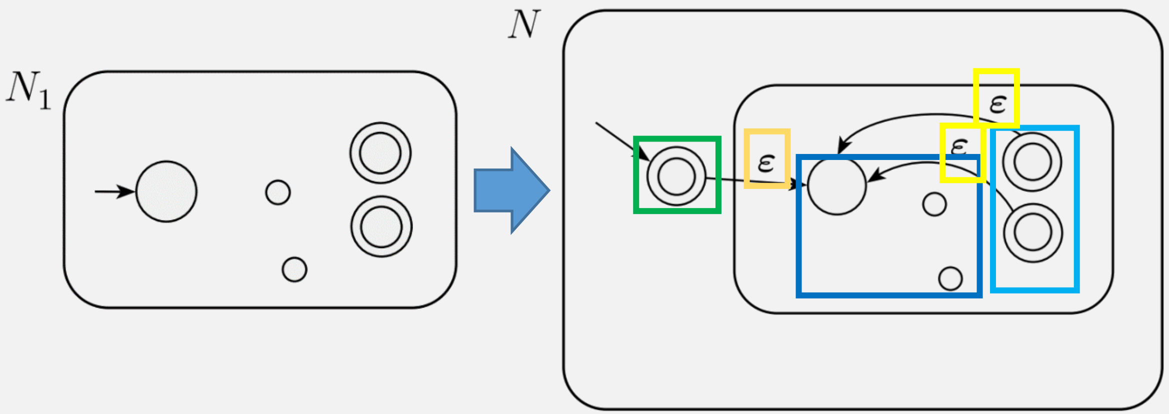

1. $Q = \{q_0\} \cup Q_1$
2. The state q_0 is the new start state.
3. $F = \{q_0\} \cup F_1$
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a)\,? & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a)\,? & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a)\,? \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} \quad ? & q = q_0 \text{ and } a = \varepsilon \\ \emptyset \quad ? & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

Next Time: Why These Closed Operations?

- Union
- Concat
- Kleene star

All regular languages can be constructed from:
- single-char strings, and
- these three combining operations!

List of Closed Ops for Reg Langs (so far)

- ☑ Union $A \cup B = \{x | \, x \in A \text{ or } x \in B\}$
- ☑ Concatentation $A \circ B = \{xy | \, x \in A \text{ and } y \in B\}$
- Kleene Star (repetition) ?

Star: $A^* = \{x_1x_2\ldots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$

Kleene Star Example

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

If $A = \{\mathtt{good}, \mathtt{bad}\}$

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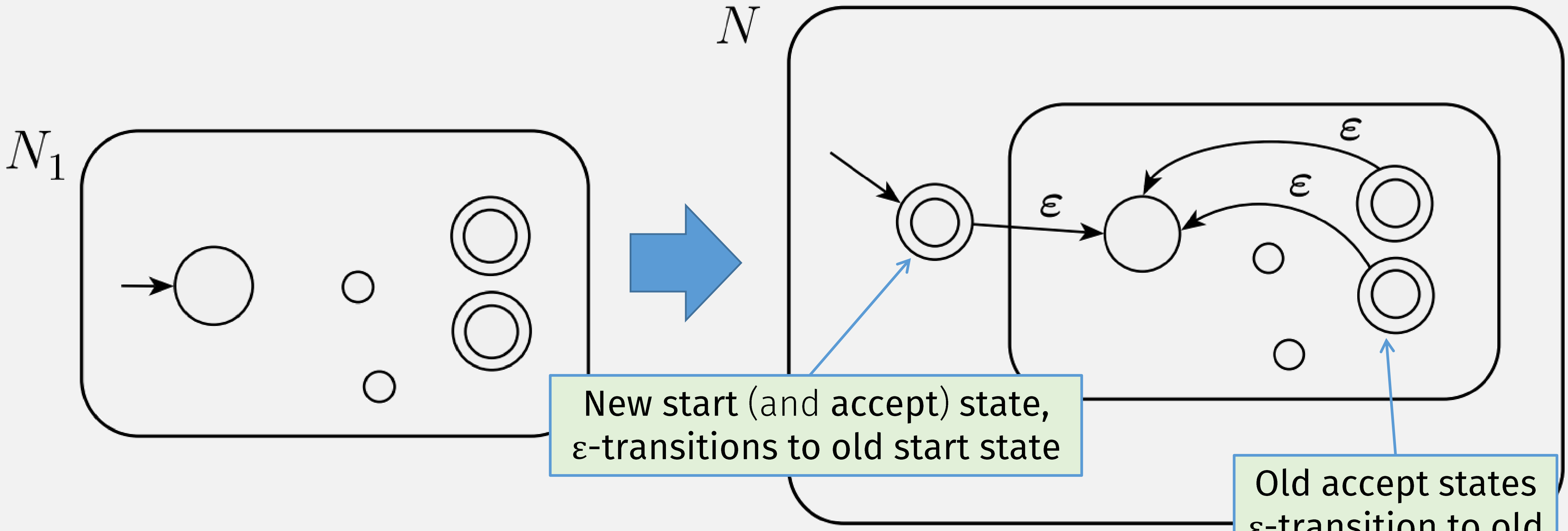
Kleene Star is Closed for Regular Langs?

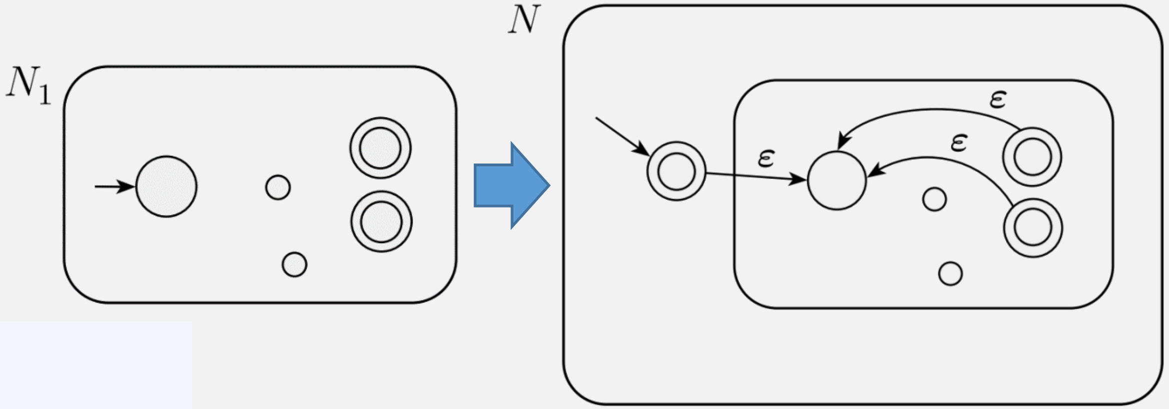


Recognizes language A

Recognizes language A^*

HW4 exercise

Kleene Star is Closed for Regular Langs



THEOREM

The class of regular languages is closed under the star operation.

Why These (Closed) Operations?

- **Union** $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- **Concatenation** $A \circ B = \{xy | x \in A \text{ and } y \in B\}$
- **Kleene star** (repetition) $A^* = \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$

All **regular languages** can be constructed **from:**

- (language of) **single-char strings** (from some alphabet)**,** and
- these **three closed operations!**