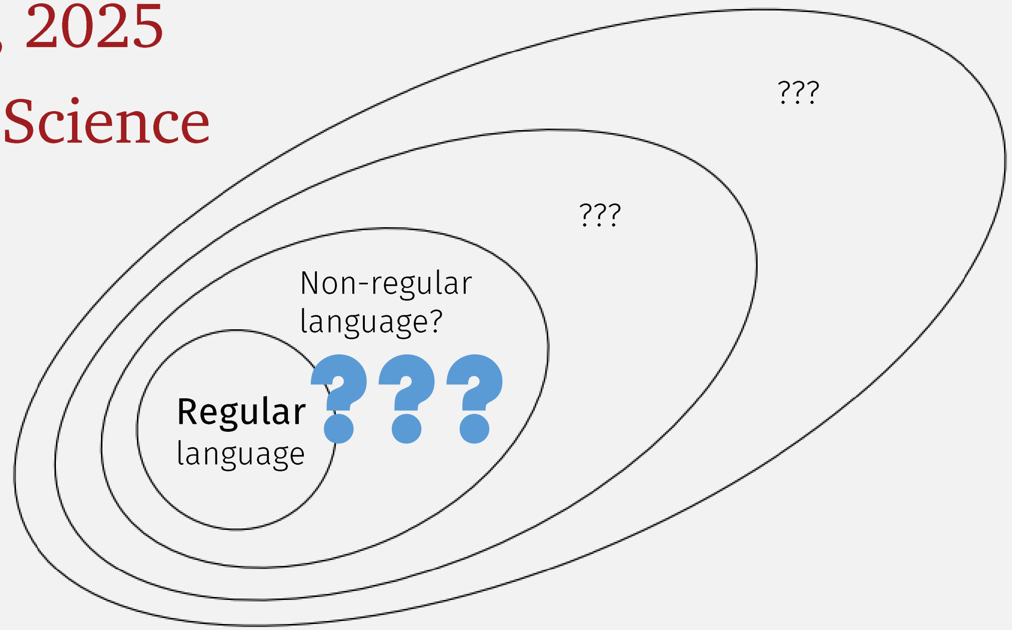
CS 420 / CS 620

Non-Regular Languages

Wednesday October 15, 2025

UMass Boston Computer Science

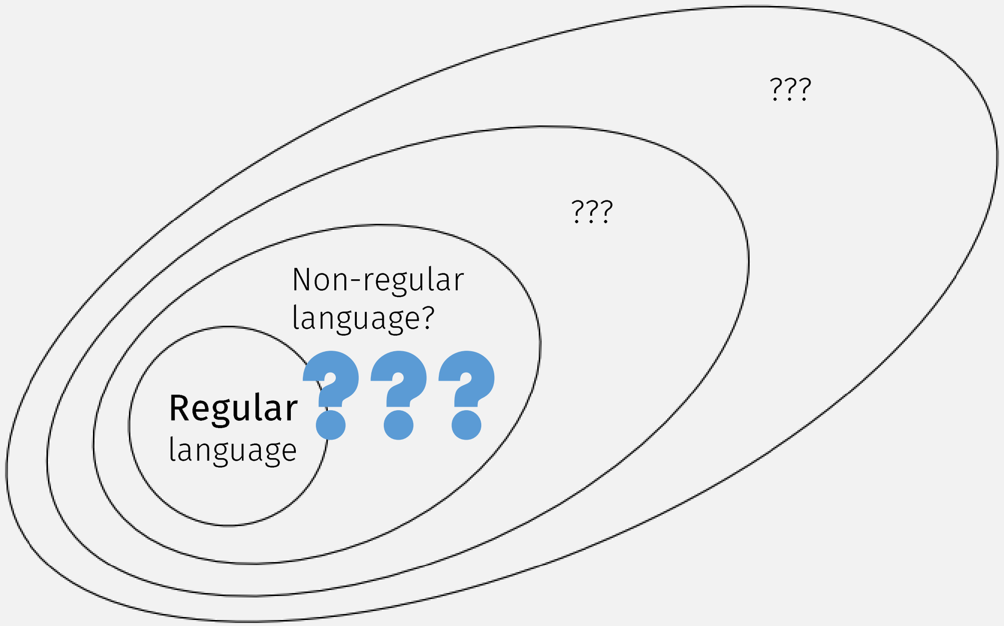


Announcements

- HW 5
  - ~~Due (unofficial): **Mon 10/13 12pm (noon)**~~
  - ~~Due (up to): **Wed 10/15 12pm (noon)**~~

- HW 6
  - Out: **Mon 10/13 12pm (noon)**
  - Due: **Wed 10/20 12pm (noon)**
  - **2 extra credit** (do 1) **+ 1 problem**

- New Office Hours Time
  - **Mon 10-11:30am (Richard Chang)**



In-class questions preview – Regular or not?

Q1.1 Regular Languages
1 Point

True or False. The Pumping Lemma can be used to prove that a language is a regular language.

Q1.2 non-Regular Languages
1 Point

True or False. The Pumping Lemma can be used to prove that a language is **not** a regular language.

A ***language*** is a set of strings.

So Far: Regular or Not?

M ***recognizes language*** A if $A = \{w \mid M \text{ accepts } w\}$

- Many ways to **prove a <u>language is regular</u>:**
  - Construct a **DFA recognizing** it (def of Regular Language)
  - Construct an **NFA recognizing** it (Sipser 1.40)
  - Create a **regular expression** describing it (Sipser 1.54)

- Bc we proved: **Regular Expression ⇔ NFA ⇔ DFA ⇔ Regular Language**

- But ***<u>not</u>* all languages** are **regular!**
  - E.g., <u>programming language syntaxes</u> are <u>not regular</u>
    - **language of all `Python` programs,** or **all HTML/XML pages,** are <u>not regular</u>
  - That means:
    - There is ***<u>no</u>*** **DFA** or **NFA** that:
      - **accepts** valid `Python` programs (and **rejects invalid ones**)
    - And, there is ***<u>no</u>*** **regular expression** that:
      - **describes all valid `Python`** or **HTML programs** (a common mistake)**!**

Someone Who Didn't Pay Atte

RegEx match open tags except XHTML self-con

Asked 10 years, 10 months ago Active 1 month ago Viewed 2.9m times

1553

6572

I need to match all of these opening tags:

```
<p>
<a href="foo">
```

But not these:

Trying to use regular expressions to describe the non-regular HTML language

4414

You can't parse [X]HTML with regex. Because HTML can't be parsed
Regex is not a tool that can be used to correctly parse HTML. As I h
HTML-and-regex questions here so many times before, the use of re
allow you to consume HTML. Regular expressions are a tool that is
sophisticated to understand the constructs employed by HTML. HTM
regular language and hence cannot be parsed by regular expressior
queries are not equipped to break down HTML into its meaningful pa
times but it is not getting to me. Even enhanced irregular regular exp
used by Perl are not up to the task of parsing HTML. You will never

Someone who paid attention in 620/420 …

HTML is a language of sufficient complexity that it cannot be parsed by regular expressions. Even Jon Skeet cannot parse HTML using regular expressions. Every time you attempt to parse HTML with regular expressions, the unholy child weeps the blood of virgins, and Russian hackers pwn your webapp. Parsing HTML with regex summons tainted souls into the realm of the living. HTML and regex go together like love, marriage, and ritual infanticide. The <center> cannot hold it is too late. The force of regex and HTML together in the same conceptual space will destroy your mind like so much watery putty. If you parse HTML with regex you are giving in to Them and their blasphemous ways which doom us all to inhuman toil for the One whose Name cannot be expressed in the Basic Multilingual Plane, he comes. HTML-plus-regexp will liquify the nerves of the sentient whilst you observe, your psyche withering in the onslaught of horror. Regex-based HTML parsers are the cancer that is killing StackOverflow *it is too late it is too late we cannot be saved* the trangession of a child ensures regex will consume all living tissue (except for HTML which it cannot, as previously prophesied) *dear lord help us how can anyone survive this scourge* using regex to parse HTML has doomed humanity to an eternity of dread torture and security holes *using regex as a tool to process* HTML establishes a breach *between this world* and the dread realm of corrupt entities (like SGML entities, but *more corrupt) a mere glimps*e of the world of rege**x parsers for HTML will ins**tantly transport a p*rogrammer's consciousness* into a w*orld* of ceaseless screaming, he comes, the pestilent slithy regex-infection wil**l devour your HT**ML parser, application and existence for all time like Visual Basic only worse *he comes he comes do not fi*ght h**e com̡e̶s, ̕h̵i**s un̨ho͞ly radiańcé destro҉ying all enli̍̈́̂̈́ghtenment, HTML tags **lea͠ki̧n͘g fr̶ǫm ̡yo͟ur eye͢s̸ ̛l̕ik͏e liq**uid pain, the song of re̸gular exp**re**ssion parsing will exti*nguish the voices of mor*tal man from the sp*here I can see it can you see ̲͚̖͔̙î̩́t̲͎̩̱͔́̋̀ it is beautiful t*he f`inal snuf`fing *o*f the lie*s of Man ALL IS LOŚ͖̩͇̗̪̏̈́T ALL I*S LOST th*e pon̷y he comes h*e c̶̮omes he comes *the ich*or permeat*es all MY FACE MY FACE ᵒh god n*o NO NOO̼**O**O NΘ stop t*he an*̶͑̾̾̅ͫ͏̙̤g͇̫͛͆̾ͫ̑͆l͖͉̗̩̳̟̍ͫͥͨe̠̅s ͎a̧͈͖r̽̾̈́͒͑e n*ot rè̑ͧ̌aͨl̘̝̙̃ͤ͂̾̆ ZA̡͊͠͝LGΌ ISͮ̂҉̯͈͕̹̘̱ TO͇̹̺ͅƝ̴ȳ̳ TH̘Ë͖́̉ ͠P̯͍̭O̚N̐Y̡ H̸̡̪̯ͨ͊̽̅̾̎Ȩ̬̩̾͛ͪ̈́̀́͘ ̶̧̨̱̹̭̯ͧ̾ͬC̷̙̲̝͖ͭ̏ͥͮ͟Oͮ͏̮̪̝͍M̲̖͊̒ͪͩͬ̚̚͜Ȇ̴̟̟͙̞ͩ͌͝S̨̥̫͎̭ͯ̿̔̀ͅ

Have you tried using an XML parser instead?

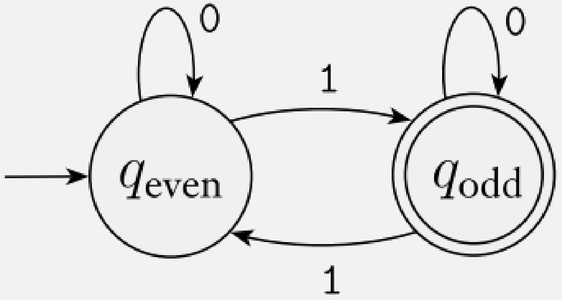
ummm … this is getting a little weird

very weird …

?????

hmm … what's this?

Flashback: Designing DFAs or NFAs



- **Each state "remembers" information about input**
  - E.g., q_{even} = "seen even # of 1s"
    q_{odd} = "seen odd # of 1s"
  - But <u>**finite**</u> **states** = <u>**finite**</u> **amount** of **info storage** (and must decide in advance)

- So <u>DFAs can't remember</u> an <u>arbitrary count</u>!
  - would require infinite states

A Non-Regular Language

An arbitrary count

L = { $\mathtt{0}^n\mathtt{1}^n$ | $n \geq 0$ }

- A DFA **recognizing** L would **require infinite states!** (impossible)
  - **States representing** zero **`0`s seen,**
  - … one **`0` seen,**
  - … two **`0`s seen** …
- This **language** represents the essence of **many PLs,** e.g., **HTML!**
  - Replace:
    - "**`0`**" with "**`<tag>`**" or "**`(`**"
    - "**`1`**" with "**`</tag>`**" or "**`)`**"

But, **how can we**
prove **non-regularness?**

- The Problem: **remembering nestedness**
  - **Need to count arbitrary nesting depths**
    - E.g., **(((…)))**
  - Thus: **most programming language syntax** is **not regular!**

Thought Exercise

Prove: Demons do not exist

???

Proving something not true is **different** (and usually harder) **than proving it true**

It's sometimes possible, but often needs new proof techniques!

We know how to: **prove** a **language** is **regular**
Can we: **prove** a **language** is **not regular**?

Quantified Logical Statements

- "**Exists**" (Existential)
  - "**Easier**" to **prove TRUE**
    - Just need **one** **example!**

$\exists x P(x)$ is true when $P(x)$ is true for at least one value of x.

"There **exists** a **natural number** n such that, $n \cdot n = 25$"

$n=5$

- "**For all**" (Universal)
  - "**Harder**" to **prove TRUE**
    - Need to **prove true for** all examples

$\forall x P(x)$ is true when $P(x)$ is true for all values of x.

"For all natural numbers n, $2 \cdot n = n + n$"

Proof by induction, on natural number n ...!

Quantified Logical Statements (this course)

- "Exists" (Existential)
  - "Easier" to prove TRUE
    - Just need one example!
  - "Harder" to prove FALSE
    - Need to prove false for all examples
- "For all" (Universal)
  - "Harder" to prove TRUE
    - Need to prove true for all examples
  - "Easier" to prove FALSE
    - Just need one (counter)example!

Language L = { … } is regular

There exists one DFA that recognizes L

Language L is not regular?

There are no possible DFAs that recognizes L

For all regular languages L, L^* is a regular language

For all strings in a regular language …

Key is finding such a statement about regular languages!

A Fact (Lemma) About Regular Languages

True for all regular languages!

∀A

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Remember: To *use* an "If *X* then *Y*" statement,
1. *prove* *X* is true,
2. *conclude* that *Y* is true

This is an "If *X* then *Y*" statement

Flashback: The Modus Ponens Inference Rule

If we know these statements are true …

- If P then Q

- P

Then we also know this statement is true …

- Q

A Lemma About Regular Languages

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

... then we can *conclude* ...

Uh ... whatever this says ...

To *use* The **Pumping lemma** for a language A ...

... first *prove* that A is a regular language ...

Q: Can we *use* The **Pumping lemma** to **prove** that a **language** is **regular?**

(but maybe **it** ***can*** **be** **used** to **prove** that a **language** is **not regular!**)

NO (but we already know many other ways to do that!)

Equivalence of Conditional Statements

- Yes or No? “If X then Y” **is equivalent to:**
  - “If Y then X” (**converse**) Seen Previously
    - No!
  - “If not X then not Y” (**inverse**)
    - No!
  - “If not Y then not X” (**contrapositive**)
    - Yes!

If-then statement

... then **the language is not regular!**

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Equivalent (**contrapositive**):
If any of these are **not** true ...

Contrapositive:
"If X then Y" is equivalent to "If **not** Y then **not** X"

Logical Inference Rules

Modus Ponens

Premises (known facts)

- If P then Q
- P is true

Conclusion (new fact)

- Q is true

Modus Tollens (contrapositive)

Premises (known facts)

- If P then Q
- Q is *not* true

Step 1: **find** a **fact that is true for all regular languages …**

Step 2: where **the fact can be proven not true!**

Conclusion (new fact)

- P is *not* true

How to: **prove** a **language** is **not regular?**

Fact About Regular Languages: Details

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Conditions are on: **strings in the language** with <u>length</u> $\geq \boldsymbol{p}$

<u>Any regular language</u> satisfies these three conditions!

<u>NOTE</u>:
- **Lemma doesn't give an exact p!**
- Only that **there is *some* string length p …**

The exact value of p differs for every regular language

The Pumping Lemma: Finite Langs

Conclusion: pumping lemma is only interesting for infinite langs! (which contain strings with repeating parts)

Lemma doesn't say what p is! Just that "there is a p …"

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

So: finite langs (specifically, all strings in the language "of length at least p") must satisfy these conditions (whatever they are)

Possible p for finite langs?

How about:
p = LENGTH(longest string) + 1

strings in the language with length ≥ p? **None!**

Therefore, all strings with length ≥ p satisfy the pumping lemma conditions! ☺

Example: a finite language {"ab", "cd"} → **ab ∪ cd**

- All finite languages are regular!
- (can easily construct DFA/NFA/Regular Expression recognizing them)

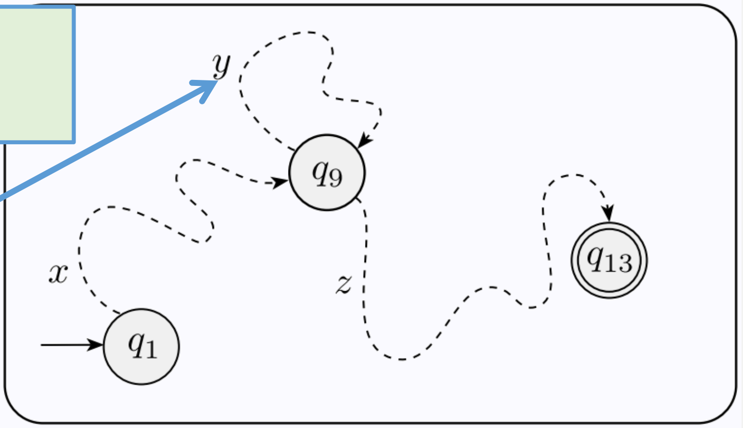
Langs With Strings With Repeatable Parts

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Lemma requires: **"pumped" string still in language!**

<u>repeatable</u> ("pumpable") **part**
(= repeatable state in DFA!)



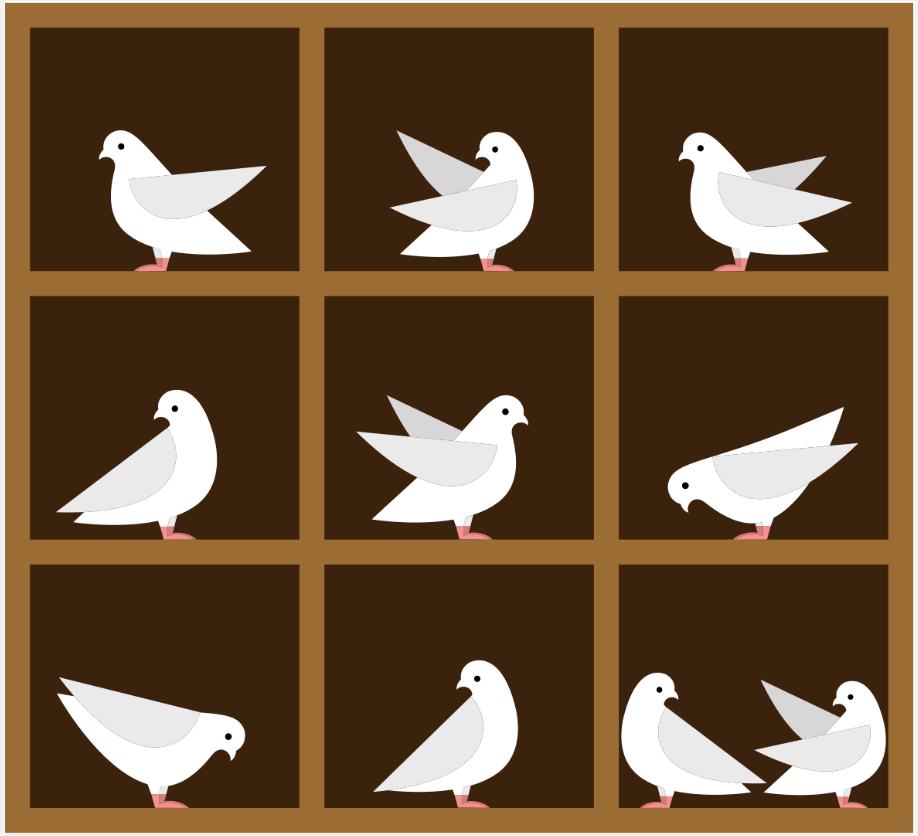
Strings with a **<u>repeatable</u> part** can be **split into 3 parts:**

- x = **part** <u>before</u> **any repeating**
- y = **<u>repeatable</u>** (or "pumpable") **part**
- z = **part** <u>after</u> **any repeating**

DFAs have finite states,
so **for "long enough"** (i.e., **length ≥** p) **inputs,**
<u>some state must repeat!</u>

e.g., **"long enough length"** = p = **# states** +1 (The Pigeonhole Principle)

The Pigeonhole Principle



If # birds > # holes, then there must be > 1 bird in some hole

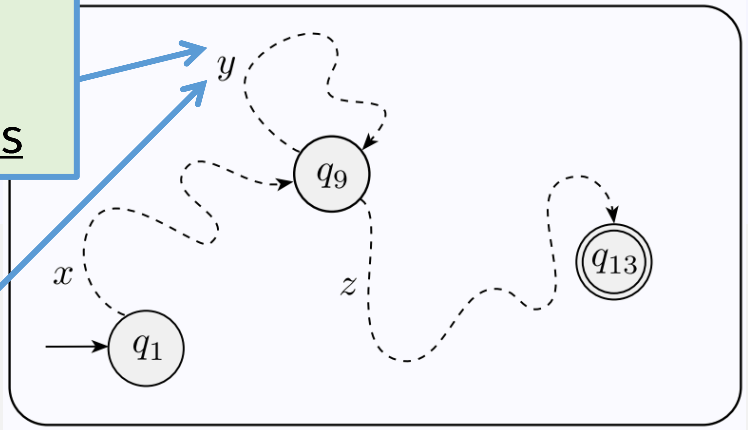
The Pumping Lemma, a Closer Look

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $x\boxed{y^i}z \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

So a **substring** that:
- *can* repeat <u>once</u>,
- *can also* be repeated <u>multiple times</u>

This is the <u>only</u> way for regular languages to have repeating patterns (KLEENE star)



In essence, **the Pumping lemma is a theorem** about <u>repeating patterns in regular languages</u>

"long enough length" = p = # **states** +1
(some state must repeat)

In-class exercise: Infinite Languages

Split the string "010" into **three parts** xyz, e.g.

$x =$ "0" $y =$ "1" $z =$ "0"

so that **<u>repeating</u>** (non-empty) y **part any number of times creates** a **new string still in** A

Now do "0110":

$x =$ "0" $y =$ "1" $z =$ "10"

<u>Example</u>: ***infinite*** **language** A = {"00", $\boxed{\text{"010"}}$, $\boxed{\text{"0110"}}$, "01110", …}

Or …?

(there could be more than one possible splitting)

The Pumping Lemma: Infinite Languages

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

"pumpable" part of string

Note: "pumpable" part cannot be empty

E.g., "010" ∈ A, so pumping lemma says it's splittable into three parts xyz, e.g.
$x = 0$, $y = 1$, $z = 0$

Example: *infinite* language A = {"00", "010", "0110" , "01110", ...}

- It's regular bc it has regular expression 01^*0

… and "pumping" (repeating) middle y part creates a string that is still in the language
- repeat once ($i = 1$): "010",
- repeat twice ($i = 2$): "0110",
- repeat three times ($i = 3$): "01110"

Pumping lemma summary:
"All infinite regular languages must have a star in its regular expression"!

Summary: The Pumping Lemma …

- … **states properties** that are true for all regular languages
- … specifically, **properties about** "long enough" **strings** in **reg. langs**
- In general, **it describes** repeating patterns in **reg. langs**

IMPORTANT:

- **The Pumping lemma** cannot prove that a **language is regular**!
- But **… we** can **use it to** prove that a **language is not regular**

Pumping lemma summary:
"All infinite regular languages must have a star in its regular expression"!

… **by showing** that the repeating pattern is not expressible with a star regular expression!

If-then statement

... then **the language is <u>not</u> regular**

Pumping lemma $\boxed{\text{If } A \text{ is a regular language, then}}$ there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Equivalent (**contrapositive**):
If **any of these are <u>not</u> true ...**

Contrapositive:
"If *X* then *Y*" is <u>equivalent</u> to "If **not** *Y* then **not** *X*"

Kinds of Mathematical Proof

- Deductive Proof
  - Logically infer (i.e., with modus ponens) conclusion from known definitions and assumptions
- Proof by induction
  - Used to prove properties of recursive definitions or functions
- Proof by contradiction
  - Proving the contrapositive

How To Do Proof By Contradiction

3 easy steps:

1. Assume: the opposite of the statement to prove

2. Show: the assumption leads to a contradiction

3. Conclude: the original statement must be true

Pumping Lemma: Non-Regularity Example

This repetition pattern cannot be expressed with a star regular expression?

Let B be the language $\{0^n 1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

... **by showing** that the **repeating pattern** is **not expressible with** a **star regular expression**!

Pumping lemma summary:
"All infinite regular languages must have a star in its regular expression"!

Want to prove: 0^n1^n **is not** a regular language

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Reminder: **Pumping lemma says: all strings $0^n1^n \geq$ length p are splittable** into xyz where y **is pumpable**

So FIND: **string $\geq$ length p** that is **not splittable** into xyz where y **is pumpable**

Proof (by contradiction):

Now we must find a contradiction ...

- Assume: 0^n1^n **is** a regular language
  - So **it must satisfy the pumping lemma**
  - I.e., **all strings $\geq$ length p are pumpable**
- Counterexample = 0^p1^p

(May take multiple trial-and-error attempts to find this)

We must show: **there is no possible way to split** this **string** to **satisfy the conditions of the pumping lemma!**

(HW requires that this **counterexample case analysis** is separate (but referred to) from the main proof)

(This should go in a Statements / Justification Table of course)

Want to prove: 0^n1^n **is not** a regular language

... then not true

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Contrapositive: If **not true** ...

Possible Split: y = all 0s

Reminder: Pumping lemma says: all strings $0^n1^n \geq$ length p **are splittable** into xyz where y is pumpable

So FIND: string $\geq$ length p that is **not splittable** into xyz where y is pumpable

Proof (by contradiction):

Contradiction??

Not yet!

- Assume: 0^n1^n **is** a regular language
  - So it must satisfy the pumping lemma
  - I.e., all strings $\geq$ length p are pumpable
- Counterexample = 0^p1^p
- Choose xyz split so y contains:
  - $\boxed{\text{all 0s}}$

$$\overbrace{00\ \ldots\ 0}^{p\ 0s}\overbrace{11\ \ldots\ 1}^{p\ 1s}$$

$$\underbrace{0}_{x}\underbrace{0\ \ldots\ 0}_{y}\underbrace{11\ \ldots\ 1}_{z}$$

BUT ... pumping lemma requires **only one** pumpable splitting

So the proof is not done!

Is there another way to split into xyz ?

- Pumping y: produces a string with **more 0s than 1s**
  - ... not in the language 0^n1^n !
  - So 0^p1^p is not pumpable? (according to pumping lemma)
  - So 0^n1^n is a not regular language? (contrapositive)
  - Is this is a **contradiction** of the assumption???

Want to prove: 0^n1^n is **not** a regular language

Possible Split: y = all 1s

Proof (by contradiction):

- Assume: 0^n1^n **is** a regular language
  - So it must satisfy the pumping lemma
  - I.e., all strings ≥ length p are pumpable
- Counterexample = 0^p1^p

$$\underbrace{\overbrace{00\ldots0}^{p\text{ 0s}}}_{x}\overbrace{\underbrace{11}_{y}\underbrace{\ldots1}_{z}}^{p\text{ 1s}}$$

- Choose xyz split so y contains:
  - all 1s

Is there another way to split into xyz ?

- Is this string pumpable (repeating y produces string still in 0^n1^n)?
  - No!
  - By the same reasoning as in the previous slide

Want to prove: 0^n1^n is **not** a regular language

Possible Split: y = 0s and 1s

Proof (by contradiction):

- Assume: 0^n1^n **is** a regular language
  - So it must satisfy the pumping lemma
  - I.e., all strings ≥ length p are pumpable
- Counterexample = 0^p1^p
- Choose xyz split so y contains:
  - both 0s and 1s

$\overbrace{00\ldots0}^{p\text{ 0s}}\overbrace{11\ldots1}^{p\text{ 1s}}$

$x \quad y \quad z$

- Is this string pumpable (repeating y produces string still in 0^n1^n)?
  - No!
  - Pumped string will have equal 0s and 1s …
  - But they will be in the wrong order: so there is still a **contradiction**!

Did we examine every possible splitting?

Yes! QED

But maybe we did't have to …

The Pumping Lemma: Condition 3

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

The repeating part y ...
must be in the first p characters!

$$\overbrace{00 \ldots 0}^{p \text{ 0s}}11 \ldots 1$$

$$\underbrace{00 \ldots 0}_{y \text{ must be in here!}}11 \ldots 1$$

The Pumping Lemma: Pumping Down

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $\boxed{i \geq 0,}$ $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Repeating part y must be non-empty ...
but can be repeated zero times!

Example: $L = \{0^i1^j \mid i > j\}$

Want to prove: $L = \{0^i1^j \mid i > j\}$ **is not** a regular language

Pumping Down

Proof (by contradiction):

contradiction

- **Assume:** L **is** a regular language
  - So it must satisfy the pumping lemma
  - I.e., all strings ≥ length p are pumpable
- Counterexample = $0^{p+1}1^p$
- Choose xyz split so y contains:
  - all 0s
  - (Only possibility, by condition 3)

$$\overbrace{00\ldots0}^{p+1\text{ 0s}}\overbrace{11\ldots1}^{p\text{ 1s}}$$

$$\underbrace{00}_{x}\underbrace{\ldots}_{y}\underbrace{011\ldots1}_{z}$$

- Repeat y zero times (**pump down**): produces string with # 0s ≤ # 1s
  - … not in the language $\{0^i1^j \mid i > j\}$
  - So $\{0^i1^j \mid i > j\}$ does not satisfy the pumping lemma
  - So it is a not regular language
  - This is a **contradiction** of the assumption!

Next Time (and rest of the Semester)

- If a language is not regular, then what is it?
- There are many more classes of languages!