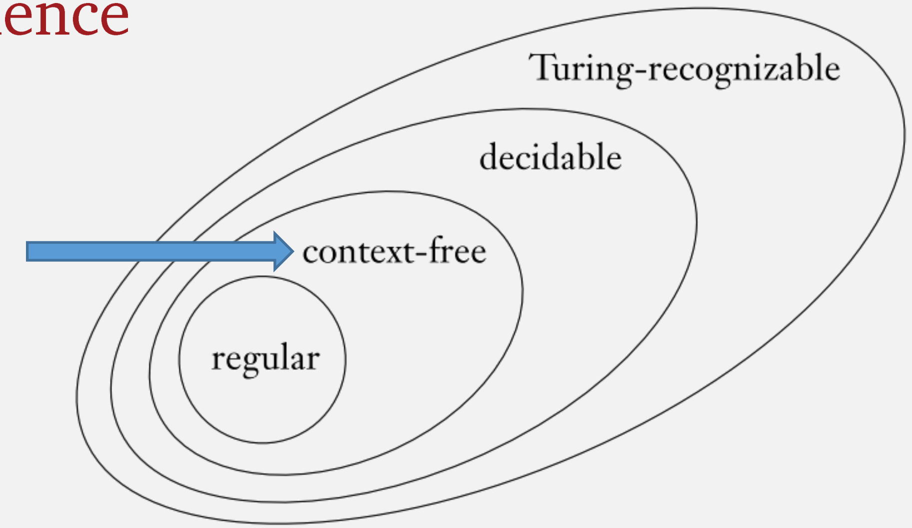
CS 420 / CS 620

Context-Free Grammars (CFGs) and Context-Free Languages (CFLs)

Monday October 20, 2025

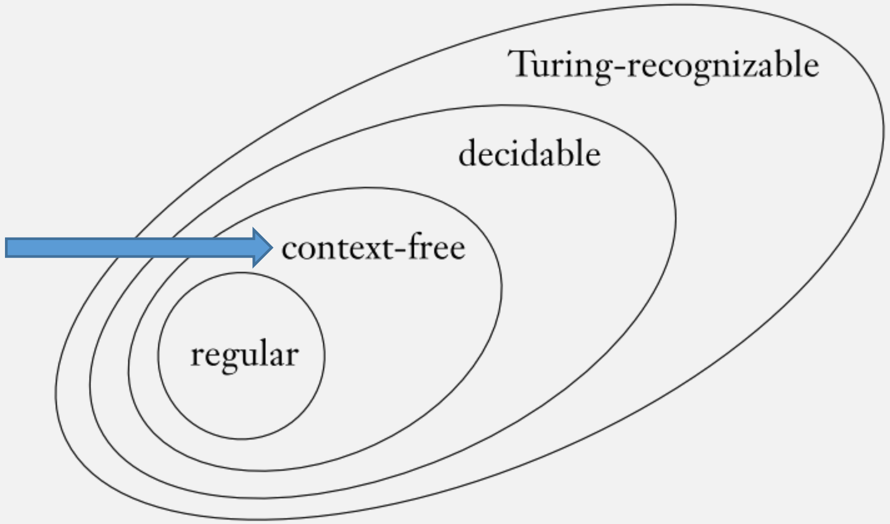
UMass Boston Computer Science



Announcements

- HW 6
  - ~~Due: Mon 10/20 12pm (noon)~~

- HW 7
  - Out: Mon 10/20 12pm (noon)
  - Due: Mon 10/27 12pm (noon)



Last Time:

Non-Regular Languages

Example:

An arbitrary count

$L = \{\, \mathtt{0}^n\mathtt{1}^n \mid n \geq 0 \,\}$

- **A DFA recognizing** L would **require infinite states!** (impossible)
  - States representing: **zero `0`s seen, one `0` seen, two `0`s, …**
- **This language is the same as many PLs,** e.g., **HTML!**
  - To better see this replace:
    - "`0`" with "`<tag>`" or "`(`"
    - "`1`" with "`</tag>`" or "`)`"
- The Problem: **remembering <u>nestedness</u>**
  - **Need to count arbitrary nesting depths**
    - E.g., **`{ { { … } } }`**
  - Thus: **most programming language syntax** is **<u>not regular</u>**!

We can *<u>prove</u>* **non-regularness** … with the **Pumping Lemma** (and **proof by contradiction**)

But … **what kind of language is it then?**

A Context-Free Grammar (CFG)

terminals

A **rule** maps:
Variable to String (of Variables and/or Terminals)

Top variable is:
Start variable

$$
\begin{aligned}
A &\rightarrow 0A1\\
A &\rightarrow B\\
B &\rightarrow \#
\end{aligned}
$$

Variables
(a.k.a., **non-terminals**)
on RHS of rules

Substitution rules
(a.k.a., **productions**)

terminals (analogous to DFA's alphabet)

Context-Free Grammar (CFG): Formally

Grammar $G_1 = (V, \Sigma, R, S)$

R is this set of rules: Var->String (of vars+terminals) mappings:

Top variable is:
Start variable

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Variables
(a.k.a., **non-terminals**)

Substitution rules
(a.k.a., **productions**)

terminals (analogous to DFA's alphabet)

CFG Practical Application:
Used to describe programming language syntax!

A ***context-free grammar*** is a 4-tuple $\boxed{(V, \Sigma, R, S)}$, where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

$V = \{A, B\}$,

$\Sigma = \{0, 1, \#\}$,

$S = A$,

Java Syntax: Described with CFGs

ORACLE

Java SE > Java SE Specifications > Java Language Specification

Chapter 2

Prev

Definition:
A **CFG describes a context-free language!**

(Could you write it in our typical **IF-THEN** form?)

Chapter 2. Grammars

This chapter describes the context-free grammars used in this specification

2.1. Context-Free Grammars

"productions" = rules

"nonterminal" = variable

A *context-free grammar* consists of a number of *productions*. Each production has an abstract symbol called a *nonterminal* as its *lef hand side*, and a sequence of one or more nonterminal and *terminal* symbols as its *right-hand side*. For each grammar, the terminal symbols are drawn from a specified *alphabet*.

"goal symbol" = Start variable

Starting from a sentence consisting of a single distinguished nonterminal, called the *goal symbol*, a given context-free grammar specifies a language, namely, the set of possible sequences of terminal symbols that can result from repeatedly replacing any nonterminal in the sequence with a right-hand side of a production for which the nonterminal is the left-hand side.

A **CFG** specifies a language!

(definition of a **language:** set of sequences of symbols)

2.2. The Lexical Grammar

A *lexical grammar* for the Java programming language is given in §3. This grammar has as its terminal symbols the characters of th Unicode character set. It defines a set of productions, starting from the goal symbol *Input* (§3.5), that describe how sequences of

https://docs.oracle.com/javase/specs/jls/se7/html/jls-2.html

Analogies

Regular Language	Context-Free Language (CFL)
Regular Expression	Context-Free Grammar (CFG)
A **Reg Expr** <u>describes</u> a **Regular lang**	A **CFG** <u>describes</u> a **CFL**

thm

def

(partially)

Python Syntax: Described with a CFG

10. Full Grammar specification

This is the full Python grammar, as it is read by the parser generator and used to parse Python source files:

```
# Grammar for Python

# NOTE WELL: You should also follow all the steps listed at
# https://devguide.python.org/grammar/

# Start symbols for the grammar:
#       single_input is a single interactive statement;
#       file_input is a module or sequence of commands read from an input file;
#       eval_input is the input for the eval() functions.
#       func_type_input is a PEP 484 Python 2 function type comment
# NB: compound_stmt in single_input is followed by extra NEWLINE!
# NB: due to the way TYPE_COMMENT is tokenized it will always be followed by a NEWLINE
single_input: NEWLINE | simple_stmt | compound_stmt NEWLINE
file_input: (NEWLINE | stmt)* ENDMARKER
eval_input: testlist NEWLINE* ENDMARKER
```

(indentation checking not expressible with CFG?)

https://docs.python.org/3/reference/grammar.html

Many Other Language (partially)

~~Python~~ Syntax: Described with a CFG

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Java Syntax: Described with CFGs

Definition:
A **CFG** describes a **context-free language**!
but **what <u>strings</u>** are **in the language?**

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Java SE > Java SE Specifications > Java Language Specification

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Chapter 2. Grammars

This chapter describes the context-free grammars used in this specification to define the lexical and syntactic structure of a program

2.1. Context-Free Grammars

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Starting from a sentence consisting of a single distinguished nonterminal, called the *goal symbol*, a given context-free grammar specifies a language, namely, the set of possible sequences of terminal symbols that can result from repeatedly replacing any nonterminal in the sequence with a right-hand side of a production for which the nonterminal is the left-hand side.

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https://docs.oracle.com/javase/specs/jls/se7/html/jls-2.html

Generating Strings with a CFG

Called "context free" because: **we replace vars** with **RHS of rule,** without **looking at surrounding chars**, i.e., **context**

In-class exercise:

Write 3 more strings that can be generated by this grammar

Definition:
A **CFG describes** a **context-free language!**
but **what strings** are **in the language?**

1st rule → $A \rightarrow 0A1$

2nd rule → $A \rightarrow B$

Last rule → $B \rightarrow \#$

"Applying a rule" = replace LHS variable with RHS sequence

At each step, *arbitrarily* choose any variable to replace, and any rule to apply

Stop when: **string is all terminals**

A CFG **generates** a string, by repeatedly applying substitution rules:

Example: $A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$

Start with: Start variable

Apply 1st rule

1st rule again

1st rule again

Apply 2nd rule

Apply last rule

Generating Strings with a CFG

Definition:
A **CFG** describes a **context-free language**!
but **what** strings are **in the language?**

$$G_1 =$$
$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Strings in CFG's language
= all possible **generated** / **derived** strings

$$\boxed{L(G_1) \text{ is } \{0^n\#1^n \mid n \geq 0\}}$$

A CFG **generates** a string, by repeatedly applying substitution rules:

Example: $A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$

This sequence of steps is called a **derivation**

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

Derivations: Formally

Let $G = (V, \Sigma, R, S)$

Single-step

$$\alpha \boxed{A} \beta \underset{G}{\Rightarrow} \alpha \boxed{\gamma} \beta$$

Where: sequence of: variables or terminals

$$\alpha, \beta \in (V \cup \Sigma)^*$$

$$\boxed{A} \in V \quad \leftarrow \text{Variable}$$

$$A \rightarrow \gamma \in R \quad \leftarrow \text{Rule}$$

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Derivations: Formally

Let $G = (V, \Sigma, R, S)$

Single-step

$$\alpha A \beta \underset{G}{\Rightarrow} \alpha \gamma \beta$$

Where:

$$\alpha, \beta \in (V \cup \Sigma)^*$$

$$A \in V \leftarrow \boxed{\text{Variable}}$$

$$A \rightarrow \gamma \in R \leftarrow \boxed{\text{Rule}}$$

Multi-step (recursively defined, on # steps)

<u>Base case</u>: $\alpha \overset{*}{\underset{G}{\Rightarrow}} \alpha$ (0 steps)

<u>Recursive case</u>: (1 or more steps)

Single step

(smaller) Recursive "call"

- **If:** $\alpha \underset{G}{\Rightarrow} \beta$ and $\beta \overset{*}{\underset{G}{\Rightarrow}} \gamma$
- **Then:** $\alpha \overset{*}{\underset{G}{\Rightarrow}} \gamma$

Formal Definition of a CFL

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

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4. $S \in V$ is the start variable.

$$G = (V, \Sigma, R, S)$$

"all possible sequences of terminal symbols …"

… "that can be **generated** with rules of grammar G"

"the **language** of a grammar G is …"

$$L(G) = \left\{ w \in \Sigma^* \mid S \overset{*}{\underset{G}{\Rightarrow}} w \right\}$$

Any language that can be generated by some context-free grammar is called a ***context-free language***

Alternatively (an easier form to use in a proof is),
IF a **language** can be **generated by some CFG,**
THEN that **language** is a **CFL**

Or IF a **CFG describes** a **language,** THEN that **language** is a **CFL**

Flashback: $\{0^n 1^n \mid n \geq 0\}$

- **Pumping Lemma** says: **not a regular language**
- It's a **context-free language!**
  - Proof?
  - Key step: **Come up with CFG describing it …**
  - <u>Hint</u>: It's similar to:

$$
\begin{aligned}
A &\to 0A1 \\
A &\to B \\
B &\to \cancel{\#}\ \varepsilon
\end{aligned}
$$

$L(G_1)$ is $\{0^n \cancel{\#} 1^n \mid n \geq 0\}$

Statements and Justifications?

Proof: $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL

Statements	Justifications
1. If a $\boxed{\text{CFG describes a language,}}$ then it is a CFL	1. Definition of CFL
2. $\boxed{\text{CFG } G_1 \text{ describes } L}$ $\begin{aligned} A &\rightarrow 0A1 \\ A &\rightarrow B \\ B &\rightarrow \varepsilon \end{aligned}$	2. (Did you come up with examples???)
3. $L = \{0^n 1^n \mid n \geq 0\}$ is a CFL	3. By Statements #1 and #2

Must be the same "P" to use modus ponens

"|" symbol = Shorthand for multiple rules with same LHS variable

A String Can Have Multiple Derivations

$$\begin{aligned} \langle \text{EXPR} \rangle &\rightarrow \boxed{\langle \text{EXPR} \rangle + \langle \text{TERM} \rangle} \mid \boxed{\langle \text{TERM} \rangle} \\ \langle \text{TERM} \rangle &\rightarrow \boxed{\langle \text{TERM} \rangle \times \langle \text{FACTOR} \rangle} \mid \boxed{\langle \text{FACTOR} \rangle} \\ \langle \text{FACTOR} \rangle &\rightarrow (\langle \text{EXPR} \rangle) \mid \boxed{\texttt{a}} \end{aligned}$$

Want to generate this string: **a + a × a**

- EXPR ⇒
- EXPR + TERM ⇒
- EXPR + TERM × FACTOR ⇒
- EXPR + TERM × a ⇒

…

RIGHTMOST DERIVATION

- EXPR ⇒
- EXPR + TERM ⇒
- TERM + TERM ⇒
- FACTOR + TERM ⇒
- **a** + TERM

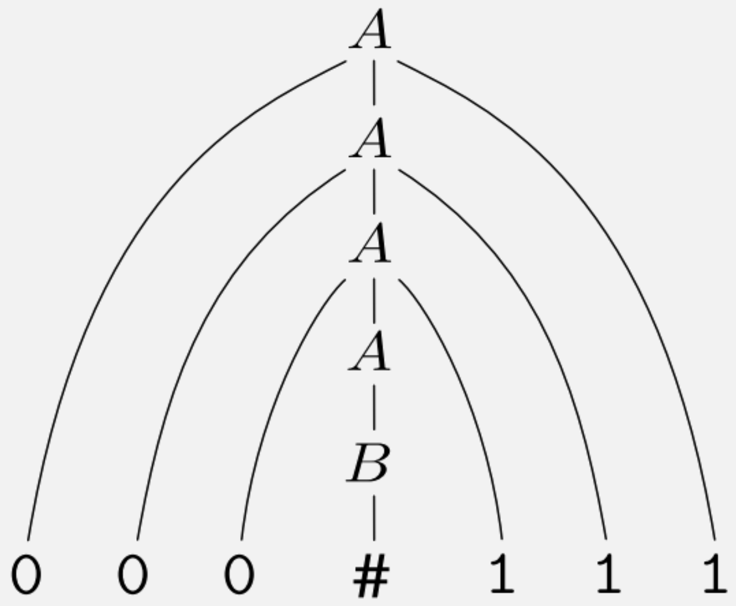
…

LEFTMOST DERIVATION

Derivations and Parse Trees

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

A derivation may also be represented as a **parse tree**



Multiple Derivations, Single Parse Tree

Leftmost deriviation

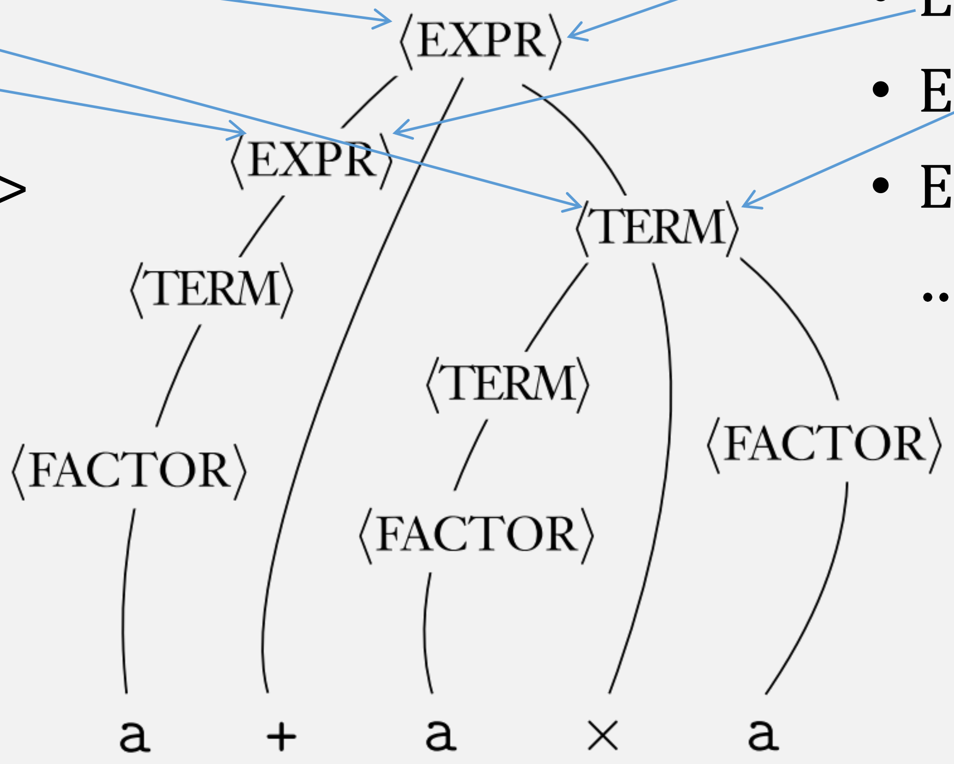
- EXPR =>
- EXPR + TERM =>
- TERM + TERM =>
- FACTOR + TERM =>
- a + TERM

…

Rightmost deriviation

- EXPR =>
- EXPR + TERM =>
- EXPR + TERM x FACTOR =>
- EXPR + TERM x a=>

…



Same parse tree

A **parse tree** represents a CFG computation … like a **sequence of states** represents a DFA computation

A Parse Tree gives **“meaning”** to a **string**

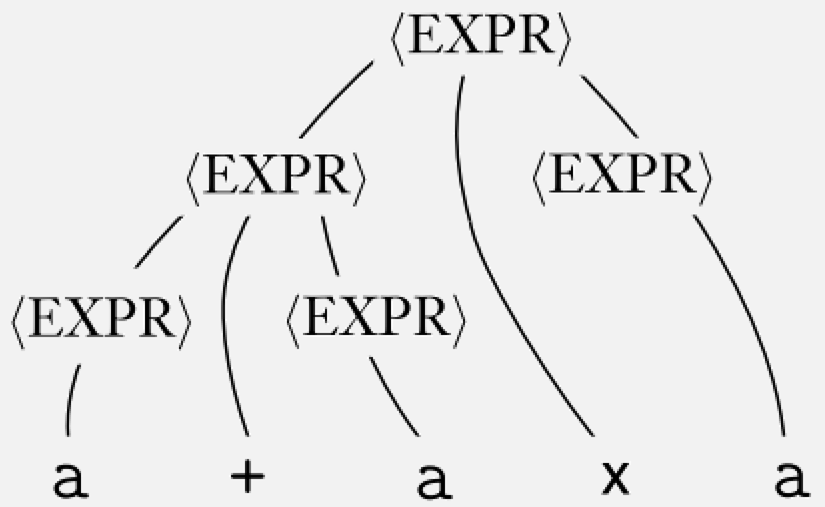
Ambiguity

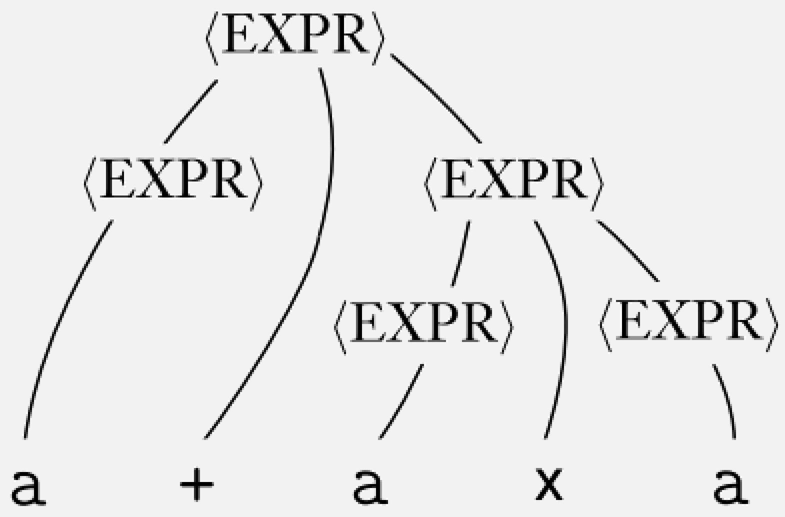
grammar G_5:

$$\langle \mathrm{EXPR} \rangle \to \langle \mathrm{EXPR} \rangle \texttt{+} \langle \mathrm{EXPR} \rangle \mid \langle \mathrm{EXPR} \rangle \texttt{x} \langle \mathrm{EXPR} \rangle \mid \texttt{(} \langle \mathrm{EXPR} \rangle \texttt{)} \mid \texttt{a}$$

Same **string,**
different **derivation,**
<u>and</u> different **parse tree!**

So this string has
<u>two</u> meanings!





Ambiguity

A string w is derived ***ambiguously*** in context-free grammar G if it has two or more different leftmost derivations. Grammar G is ***ambiguous*** if it generates some string ambiguously.

An **ambiguous grammar** can give a **string** **multiple meanings**, ie **represent** **two different computations!**
(why is this **bad**?)

Real-life Ambiguity (“Dangling” `else`)

- What is the result of this `C` program?

```
if (1) if (0) printf("a"); else printf("2");
```

```
if (1)
  if (0)
    printf("a");
  else
    printf("2");
```

VS

```
if (1)
  if (0)
    printf("a");
else
  printf("2");
```

This **string** has **<u>2 parsings</u>**, and **thus <u>2 meanings</u>!**

Ambiguous grammars are confusing. A <u>computation</u> (represented by a string) should ideally have only <u>one possible result</u>.

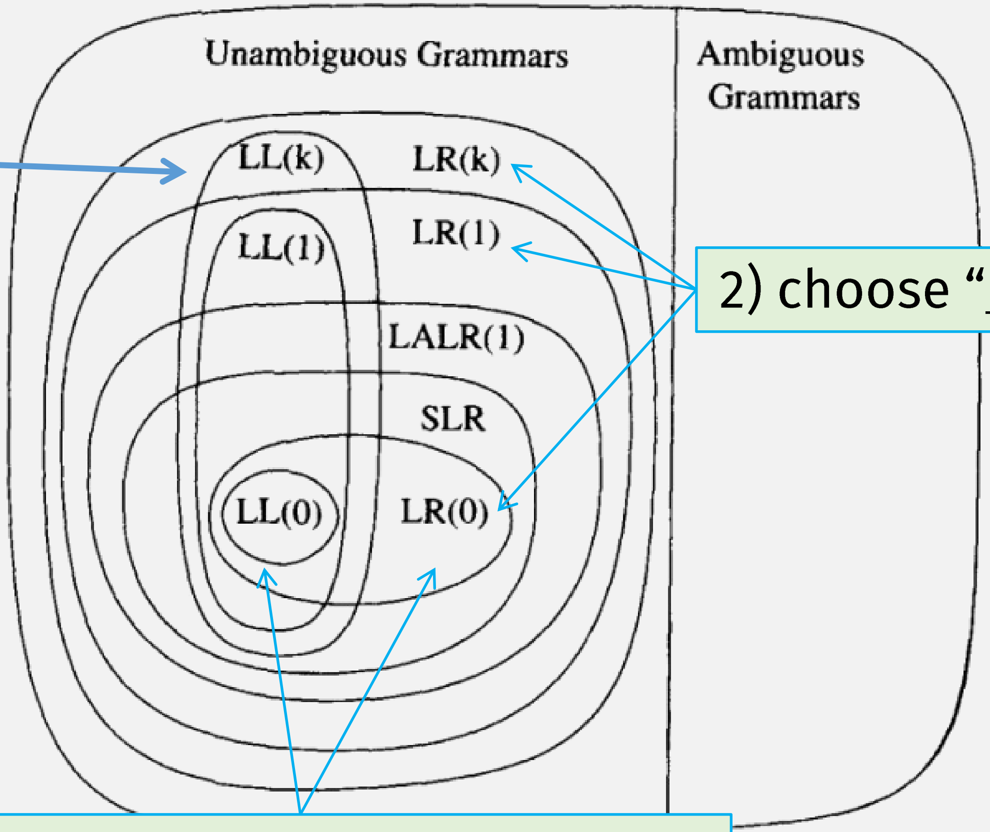
Thus <u>in practice</u>, we typically focus on the **unambiguous** <u>subset</u> of CFGs (CFLs) (more on this later)

Problem is, there’s no easy way to create an **unambiguous** grammar (it’s up to language designers to “be careful”)

Subclasses of CFLs

DCFLs

Programming language parsers / compilers are ideally in here



2) choose "look ahead" amount

2 parser design decisions:
1) Parse from left (L),
or from right (R)

All CFLS

Designing Grammars : Basics

1. **Think about what you want to “link” together**

- E.g., $0^n 1^n$
  - $A \rightarrow 0A1$
  - **# 0s** and **# 1s** are **“linked”**

- E.g., **HTML**
  - $\mathrm{ELEMENT} \rightarrow <\mathrm{TAG}>\mathrm{CONTENT}</\mathrm{TAG}>$
  - **Start** and **end tags** are **“linked”**

2. **Start** with **small grammars** (computation) and **then combine**
   - just like with **DFA**s, **NFA**s, and **programming**!

In-class exercise: Creating CFG

alphabet Σ is {0,1}

$$L = \{ w \mid w \text{ starts and ends with the same symbol} \}$$

1) come up with examples: In the language: **010, 101, 11011** **1, 0 ?** ☑

Not in the language: **10, 01, 110** **ε ?** ☒

2) Create CFG:

Needed Rules:

$S \rightarrow 0M0 \mid 1M1 \mid 0 \mid 1$ "start/end symbol are "linked" (ie, same); middle can be anything"

$M \rightarrow MT \mid \varepsilon$ "middle: all possible terminals, repeated (ie, all possible strings)"

$T \rightarrow 0 \mid 1$ "all possible terminals"

3) Check CFG: generates examples in the language; doesn't generate examples not in language

Examples Table!
(justifies that the "CFG describes L")

Designing Grammars: Building Up

- **Start** with **small grammars** and then **combine** (just like **programming**)
    - To create a **grammar for the language** $\{0^n1^n | n \geq 0\} \cup \{1^n0^n | n \geq 0\}$
    - First create **grammar for lang** $\{0^n1^n | \, n \geq 0\}$:
      $$S_1 \rightarrow 0S_11 \mid \varepsilon$$
    - Then create **grammar for lang** $\{1^n0^n | \; n \geq 0\}$:
      $$S_2 \rightarrow 1S_20 \mid \varepsilon$$
    - Then **combine:** $S \rightarrow S_1 \mid S_2$
      $$S_1 \rightarrow 0S_11 \mid \varepsilon$$
      $$S_2 \rightarrow 1S_20 \mid \varepsilon$$

New start variable and rule combines two smaller grammars

"|" = "or" = union (combines 2 rules with same left side)

(Closed) Operations for CFLs?

- **Start** with **small grammars** and then **combine** (just like **programming**)
- **"Or":** $S \to S_1 \mid S_2$
- **"Concatenate":** $S \longrightarrow S_1 S_2$
- **"Star"** (repetition)**:** $S' \longrightarrow S' S_1 \mid \varepsilon$

Could you write out the full proof?

Regular Language vs CFL Comparison

Regular Languages	Context-Free Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
describes a Regular Lang	describes a CFL

Regular Language vs CFL Comparison

Regular Languages	Context-Free Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
describes a **Regular Lang**	describes a **CFL**
Finite State Automaton (FSM)	???
recognizes a **Regular Lang**	recognizes a **CFL**

Regular Language vs CFL Comparison

	Regular Languages	Context-Free Languages (CFLs)	
thm	Regular Expression	Context-Free Grammar (CFG)	def
	describes a Regular Lang	describes a CFL	
def	Finite State Automaton (FSM)	**Push-down Automata** (PDA)	thm
	recognizes a Regular Lang	recognizes a CFL	

Regular Language vs CFL Comparison

	Regular Languages	**Context-Free Languages (CFLs)**	
thm	Regular Expression	Context-Free Grammar (CFG)	def
	describes a **Regular Lang**	describes a **CFL**	
def	Finite State Automaton (FSM)	**Push-down Automata** (PDA)	thm
	recognizes a **Regular Lang**	recognizes a **CFL**	
	Proved:	To prove:	
	Regular Lang ⇔ Regular Expr	CFL ⇔ PDA	