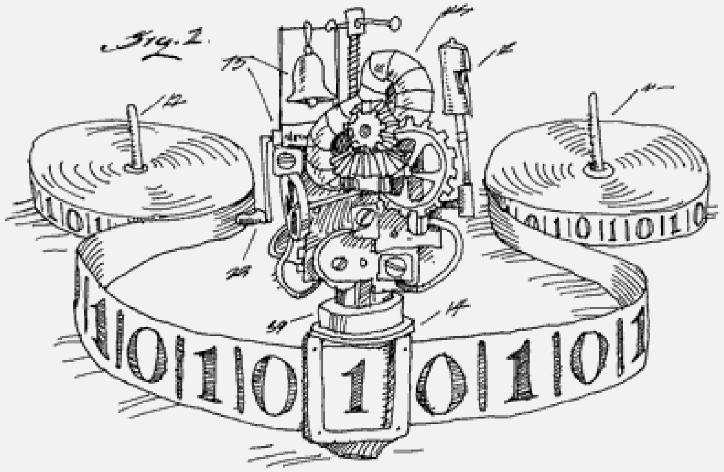
# CS 420 / CS 620

# Turing Machines (TMs)

Wed October 29, 2025

UMass Boston Computer Science



# Announcements

- HW 8
  - Out: **Mon 10/27 12pm (noon)**
  - Due: **Mon 11/3 12pm (noon)**

# In-class questions (in Gradescope)

**Q1 TM possible results**
**2 Points**

**Q1.1 TM number of possible results**
**1 Point**

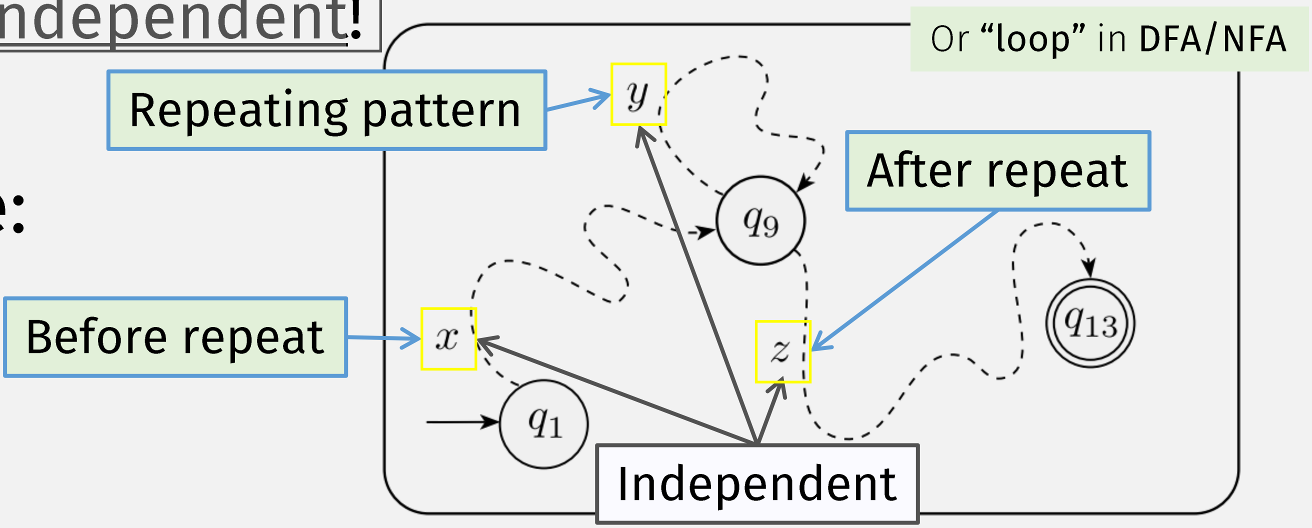
When a Turing Machine (TM) starts running with an input string, how many different possible results can there be.

**Q1.2 TM computation results**
**1 Point**

What are the possible results when a TM is run with a string input?

# *Flashback:* Pumping Lemma for Regular Langs

- **Pumping Lemma** explains how strings repeat
- **Regular language** strings repeat using **Kleene star** operation
  - Key: 3 substrings $x\,y\,z$ independent!




- A non-regular language:

$$\{0^n 1^n \mid n \geq 0\}$$

Kleene star can't express this pattern:
2nd part depends on (length of) 1st part

- Q: How do CFLs repeat?

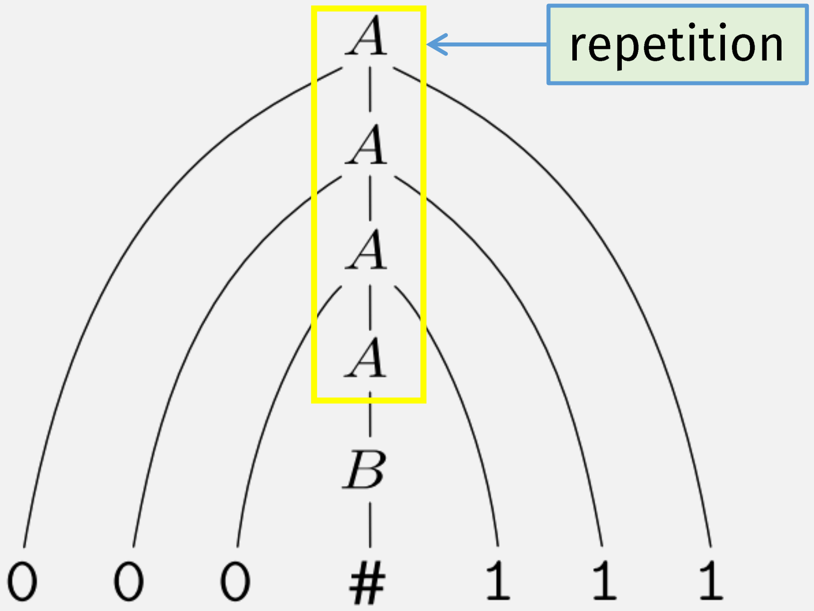
# Repetition and Dependency in CFLs

**Parts** before/after **repetition point linked** (not independent)

Repetition

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

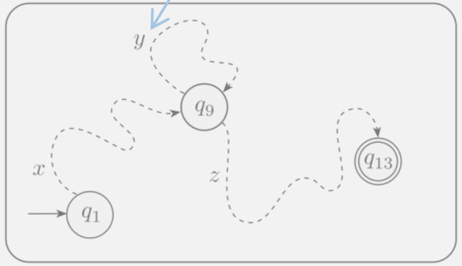
$$\{0^n\#1^n \mid n \geq 0\}$$



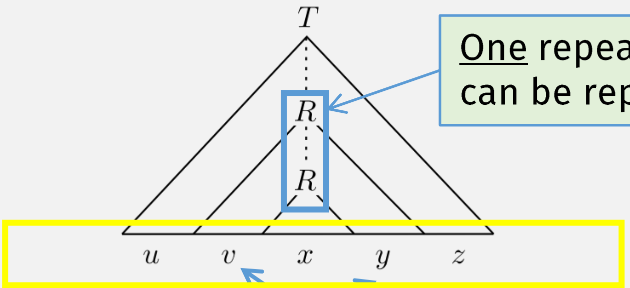

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

# How Do Strings in CFLs Repeat?

NFA can take loop transition(s) any number of times, to process repeated $y$ in input



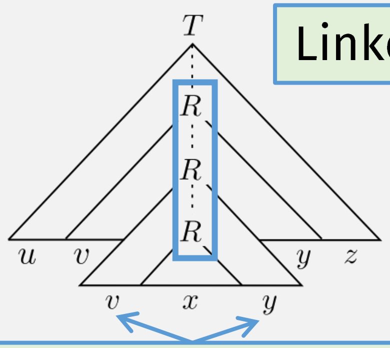

- Strings in regular languages <u>repeat states</u>
- **Strings** in **CFLs** <u>**repeat subtrees**</u> in the **parse tree**

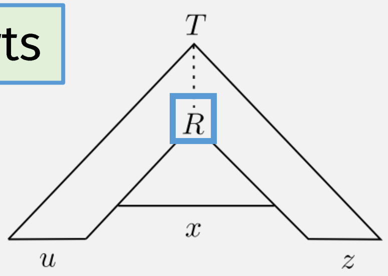



<u>One</u> repeated subtree means that it can be repeated <u>any</u> number of times

5 substrings

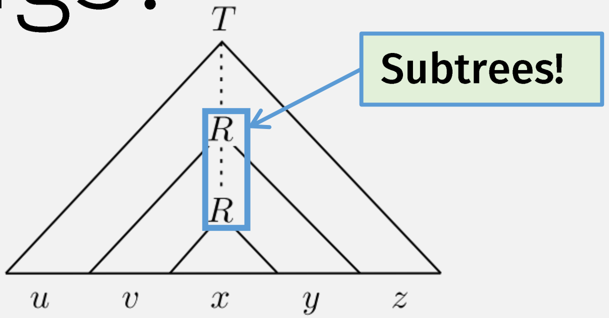
Linked parts



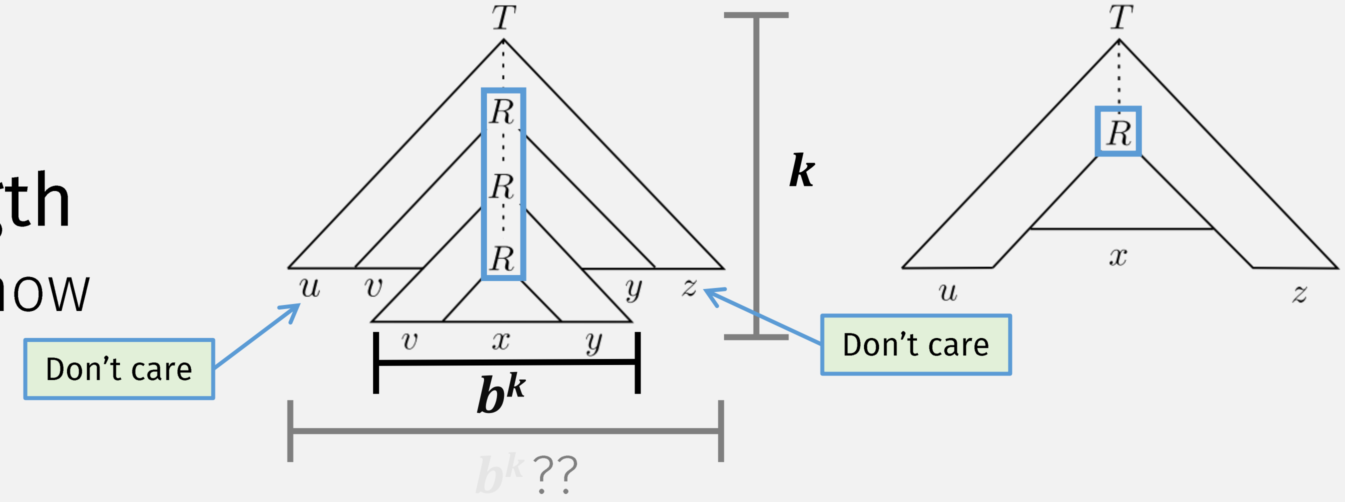

Linked parts repeat together

# Repeating Pattern in CFL Strings?




- When are we <u>guaranteed</u> **to have a repeated subtree?**
  - When <u>**height**</u> **of parse tree** > # **of rules!**

- Let ***k*** = # **of rules**
  and ***b*** = **longest rule RHS length**
  - Then **length string** where we know there's a **repeated rule** is … $b^k$
  - I.e., "**pumping length**" $p$ = $\boldsymbol{b^k}$???




**Pumping lemma for context-free languages** If $A$ is a context-free language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into five pieces $s = uvxyz$ satisfying the conditions

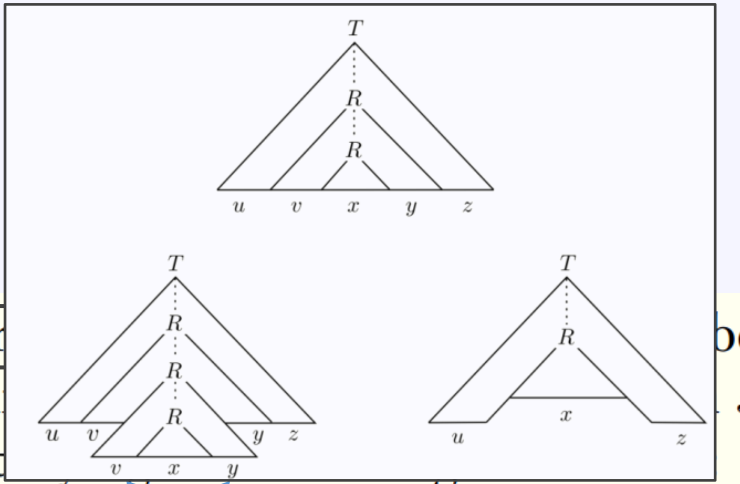
1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

# Pumping Lemma for CFLS

**Pumping lemma for context-free languages** If $A$ is a context-free language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into five pieces $s = uvxyz$ satisfying the conditions

Two pumpable parts.
But they must be pumped together!

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.




**Pumping lemma** If $A$ is a regular … ber $p$ (the pumping length) where if $s$ is any stri… $s$ may be divided into three pieces, $s = xyz$, sat…

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

One pumpable part

Two pumpable parts, pumped together

*Previously*

# A Non CFL example

$$\text{language } B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\} \text{ is not context free}$$

Intuition

- **Strings in CFLs** can have **two parts** that are **"pumped" together**
- **Language** $B$ **requires three parts** to be **"pumped" together**
- So **it's not a CFL!**

Proof?

Want to prove: $a^n b^n c^n$ **is not** a CFL

**Pumping lemma for context-free languages** If $A$ is a context-free language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Reminder: CFL Pumping lemma says: all strings $a^n b^n c^n \geq$ length $p$ are splittable into $uvxyz$ where $v$ and $y$ are pumpable

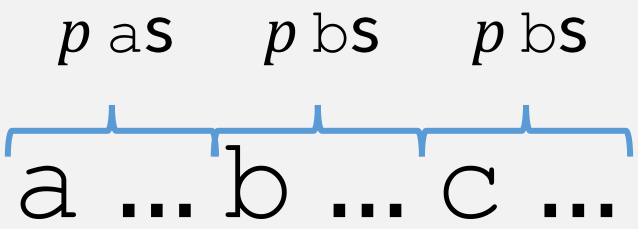
Proof (by contradiction):

Now we must find a contradiction …

- Assume: $a^n b^n c^n$ **is a CFL**
  - So it must satisfy the pumping lemma for CFLs
  - I.e., strings in lang $\geq$ length $p$ are pumpab
- Counterexample = $a^p b^p c^p$

Contradiction if:
- A string in the language ☑
- $\geq$ length $p$ ☑
- Is **not splittable** into $uvxyz$ where $v$ and $y$ are pumpable

???



Want to prove: $a^n b^n c^n$ **is not** a CFL

# Possible Splits

**Pumping lemma for context-free languages** If $A$ is a context-free language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
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Reminder: CFL Pumping lemma says: all strings $a^n b^n c^n \geq$ length $p$ are splittable into $uvxyz$ where $v$ and $y$ are pumpable

contradiction

Proof (by contradiction):

- Assume: $a^n b^n c^n$ **is** a CFL
  - So it must satisfy the pumping lemma for CFLs
  - I.e., all strings $\geq$ length $p$ are pumpable
- Counterexample = $a^p b^p c^p$

Contradiction if:

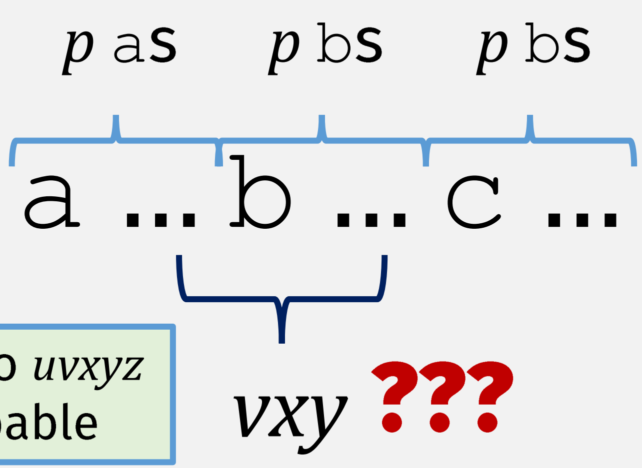
- A string in the language
- $\geq$ length $p$
- Is **not splittable** into $uvxyz$ where $v$ and $y$ are pumpable

Assumption is false

Not pumpable

- Possible Splits (using condition # 3: $|vxy| \leq p$)
  - ☒ $vxy$ is all as
  - ☒ $vxy$ is all bs
  - ☒ $vxy$ is all cs
  - ☒ $vxy$ has as and bs
  - ☒ $vxy$ has bs and cs
  - ($vxy$ cannot have as, bs, and cs)

$a^p b^p c^p$ **cannot be split** into $uvxyz$ where $v$ and $y$ are pumpable




So $a^n b^n c^n$ **is not a CFL**

# Another Non-CFL $D = \{ww \mid w \in \{0,1\}^*\}$

Be careful when choosing **counterexample** $s$: $0^p10^p1$

This $s$ **can be pumped** according to **CFL pumping lemma:**

$$\overbrace{\underbrace{000\cdots000}_{u}\ \underbrace{0}_{v}\ \underbrace{1}_{x}}^{0^p1}\ \overbrace{\underbrace{0}_{y}\ \underbrace{000\cdots0001}_{z}}^{0^p1}$$

**Pumping $v$ and $y$** (together) **produces string still in $D$!**

- **CFL Pumping Lemma conditions:**
  - ☑ **1.** for each $i \geq 0$, $uv^ixy^iz \in A$,
  - ☑ **2.** $|vy| > 0$, and
  - ☑ **3.** $|vxy| \leq p$.

No contradiction!

So **this attempt** to **prove** that the **language** is **not a CFL failed**.
(It doesn't prove that the language is a CFL!)

# Another Non-CFL $D = \{ww|\ w \in \{0,1\}^*\}$

- Need <u>another counterexample</u> string *s*:

If *vyx* is contained in **first or second half,** then **any pumping will break the match** ✖

$$0^p 1^p 0^p 1^p$$

So *vyx* must **straddle the middle**

But **any pumping still breaks the match** because **order is wrong** ✖

- **CFL Pumping Lemma conditions:**

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Now we have proven that **this language is not a CFL!**

# A Practical Non-CFL

- **XML**
  - ELEMENT → <TAG>CONTENT</TAG>
  - Where TAG **is any string**

- XML also looks like this non-CFL: $D = \{ww |\ w \in \{0,1\}^*\}$

- This means **XML is not context-free!**
  - Note: **HTML** *is* **context-free** because …
  - … there are only a **finite number of tags,**
  - so **they can be embedded** into **a finite number of rules.**

In practice:

- **XML is parsed** as a **CFL,** with a **CFG**
- Then **matching tags checked in a 2nd pass with a more powerful machine** …

# *Next:* A More Powerful Machine …

$M_1$ accepts its input if it is in language: $B = \{w\#w \mid w \in \{0{,}1\}^*\}$

$M_1$ = "On input string $w$:

1. Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do not, or if no # is found, *reject*. Cross off symbols as they are checked to keep track of which symbols correspond.

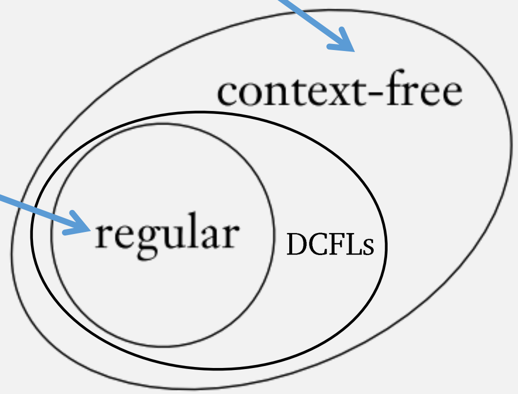
**Infinite memory** (initial contents are the **input string**)

**Can move to,** and **read/write from** <u>arbitrary</u> **memory locations!**

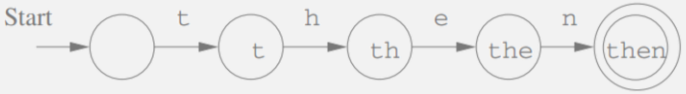
# Where We've Been, Where We're Going

- **PDAs**: recognize **context-free languages**
  - **Memory**: states + infinite **stack** (push/pop only)
  - Can't express: arbitrary dependency,
    - e.g., $\{ww \mid w \in \{0,1\}^*\}$

$A \to 0A1$
$A \to B$
$B \to \#$




- **DFAs / NFAs**: recognize **regular langs**
  - **Memory**: finite states
  - Can't express: **dependency**
    e.g., $\{0^n1^n \mid n \geq 0\}$



# Where We've Been, Where We're Going

- **Turing Machines** (TMs)
  - <u>Memory</u>: states + infinite **tape**, (arbitrary read/write)
  - Expresses any "computation"
- **PDAs**: recognize **context-free languages**
  - <u>Memory</u>: states + infinite **stack** (push/pop only)
  - Can't express: <u>arbitrary</u> dependency,
    - e.g., $\{ww \mid w \in \{0,1\}^*\}$

$A \rightarrow 0A1$
$A \rightarrow B$
$B \rightarrow \#$

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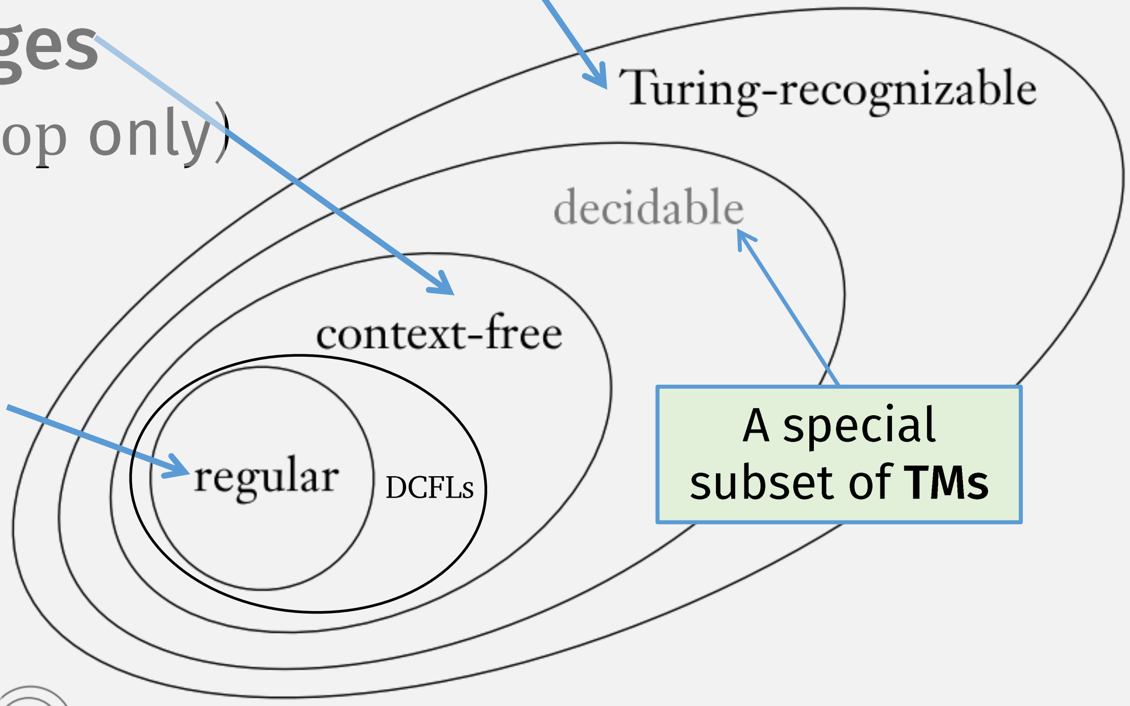
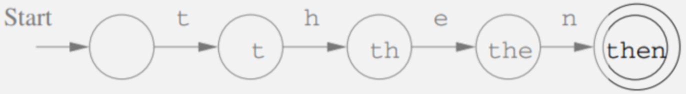


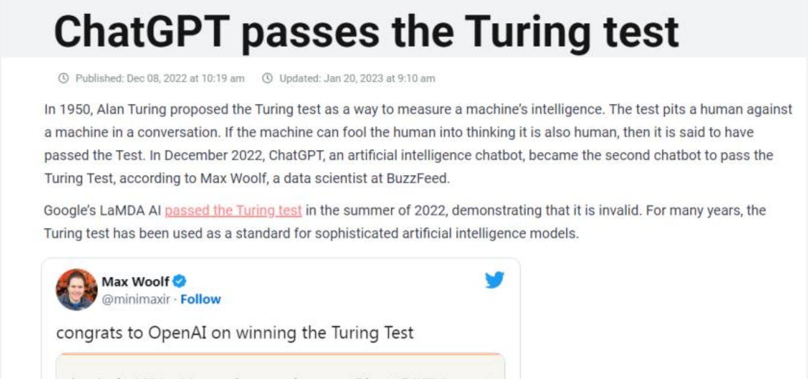



# Alan Turing

- First to **formalize a model of computation**
  - I.e., **he invented many of the ideas in this course!**

- And **worked as a codebreaker during WW2**

- Also **studied Artificial Intelligence**
  - **The Turing Test**



## ChatGPT passes the Turing test

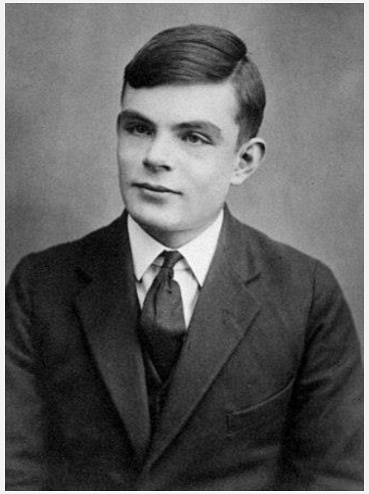
Published: Dec 08, 2022 at 10:19 am   Updated: Jan 20, 2023 at 9:10 am

In 1950, Alan Turing proposed the Turing test as a way to measure a machine's intelligence. The test pits a human against a machine in a conversation. If the machine can fool the human into thinking it is also human, then it is said to have passed the Test. In December 2022, ChatGPT, an artificial intelligence chatbot, became the second chatbot to pass the Turing Test, according to Max Woolf, a data scientist at BuzzFeed.

Google's LaMDA AI passed the Turing test in the summer of 2022, demonstrating that it is invalid. For many years, the Turing test has been used as a standard for sophisticated artificial intelligence models.

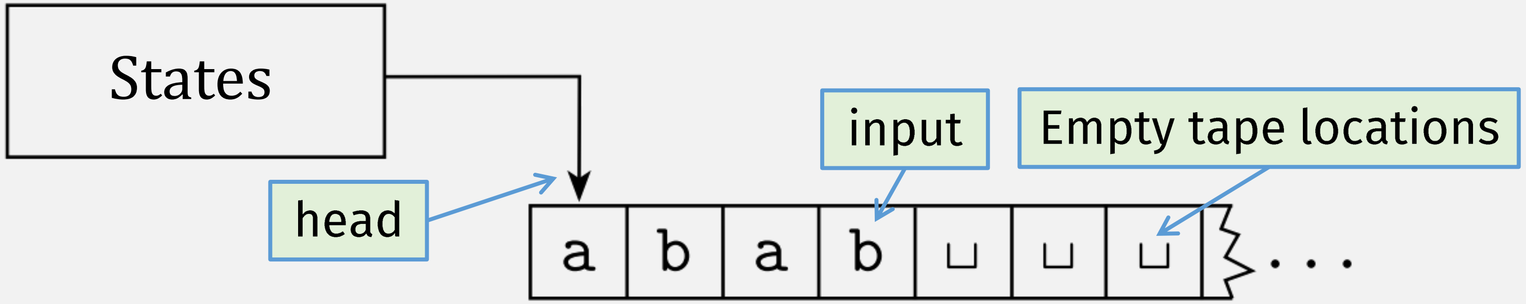
Max Woolf
@minimaxir · Follow

congrats to OpenAI on winning the Turing Test



# Finite Automata vs Turing Machines

- **Turing Machines** can read and write to arbitrary "tape" cells
  - Tape initially contains **input string**
- **Tape is infinite**
  - To the right




- Each step: **"head"** can **move left** or **right**
- **Turing Machine** can **accept** / **reject** at **any time**

Call a language ***Turing-recognizable*** if some Turing machine recognizes it.

# Turing Machine Example

TM

Define: $M_1$ **accepts inputs** in **language** $B = \{w\#w \mid w \in \{0,1\}^*\}$

Example input

tape

head

0 1 1 0 0 0 # 0 1 1 0 0 0 ␣ ...

$M_1$ = "On input string $w$:

**1.** Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do not, or if no # is found, *reject*. Cross off symbols as they are checked to keep track of which symbols correspond.

**High-level**: "Cross off"
Low-level $\delta$: write "x" char

This is a **high-level TM description**

It is **equivalent** to (but **more concise** than) our typical **(low-level) tuple descriptions**, i.e., **one step = maybe multiple $\delta$ transitions**

Analogy
"High-level": `Python`
"Low-level": `assembly language`

# Turing Machine Example

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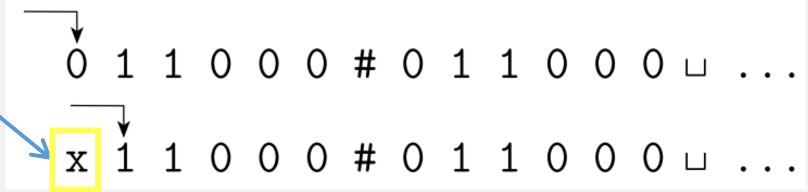

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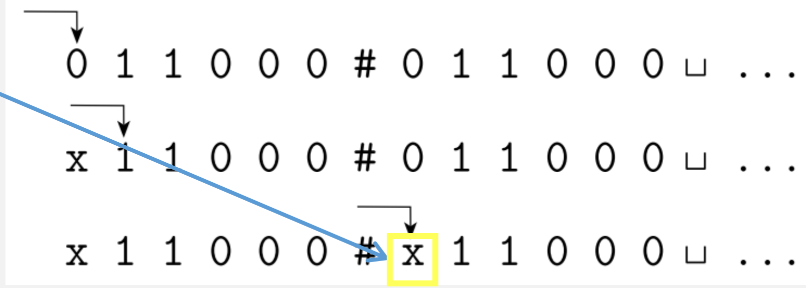

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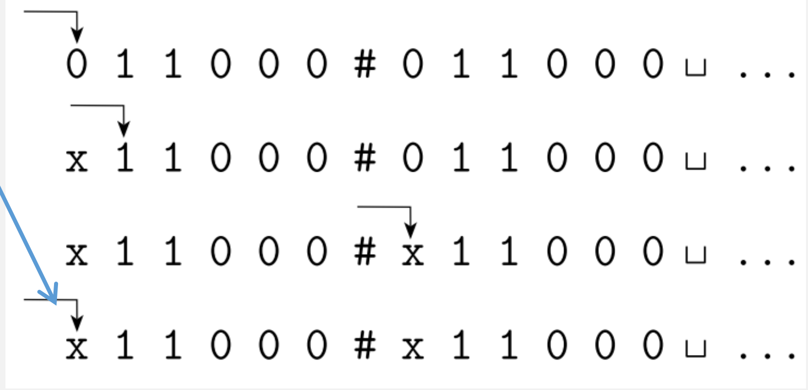
Head "zags" back to start



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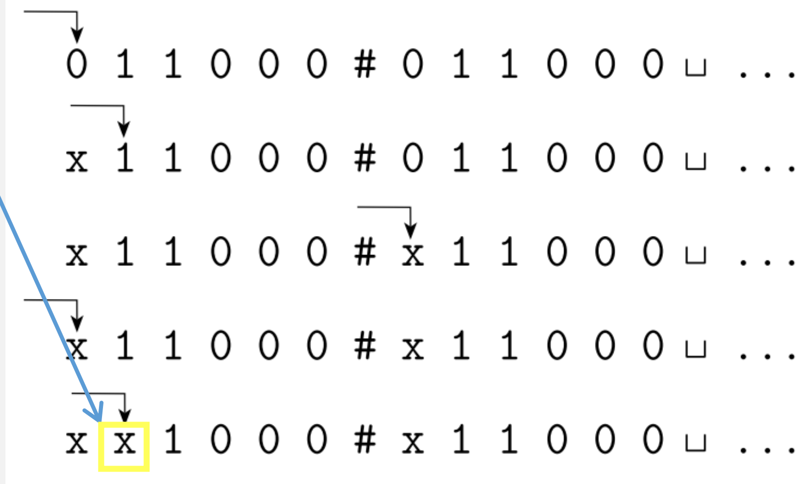
Continue crossing off



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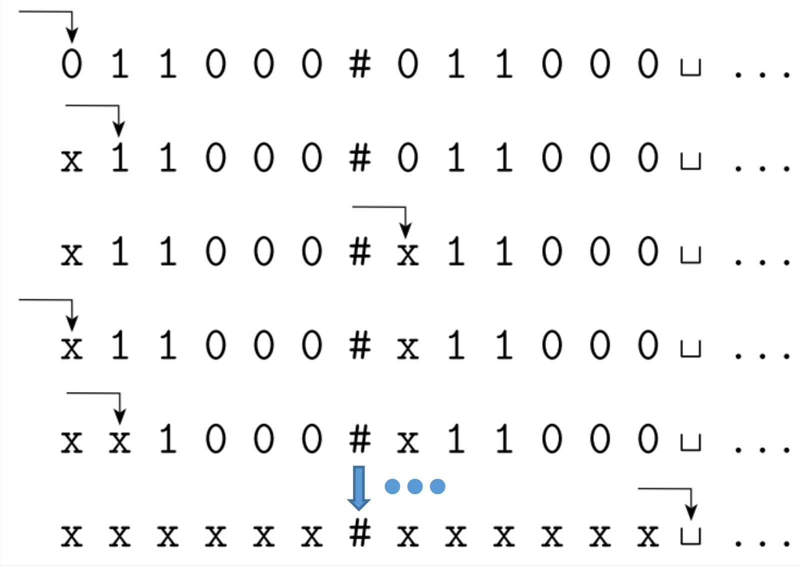
2. When all symbols to the left of the # have been crossed off, check for any remaining symbols to the right of the #. If any symbols remain, *reject*; otherwise, *accept*."



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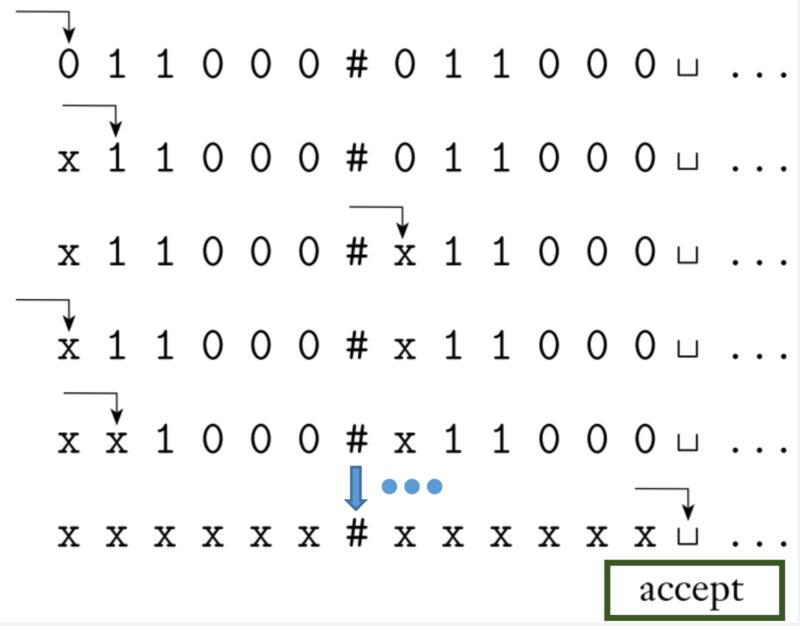


# Turing Machines: Formal Definition

This is a "**low-level**" TM description

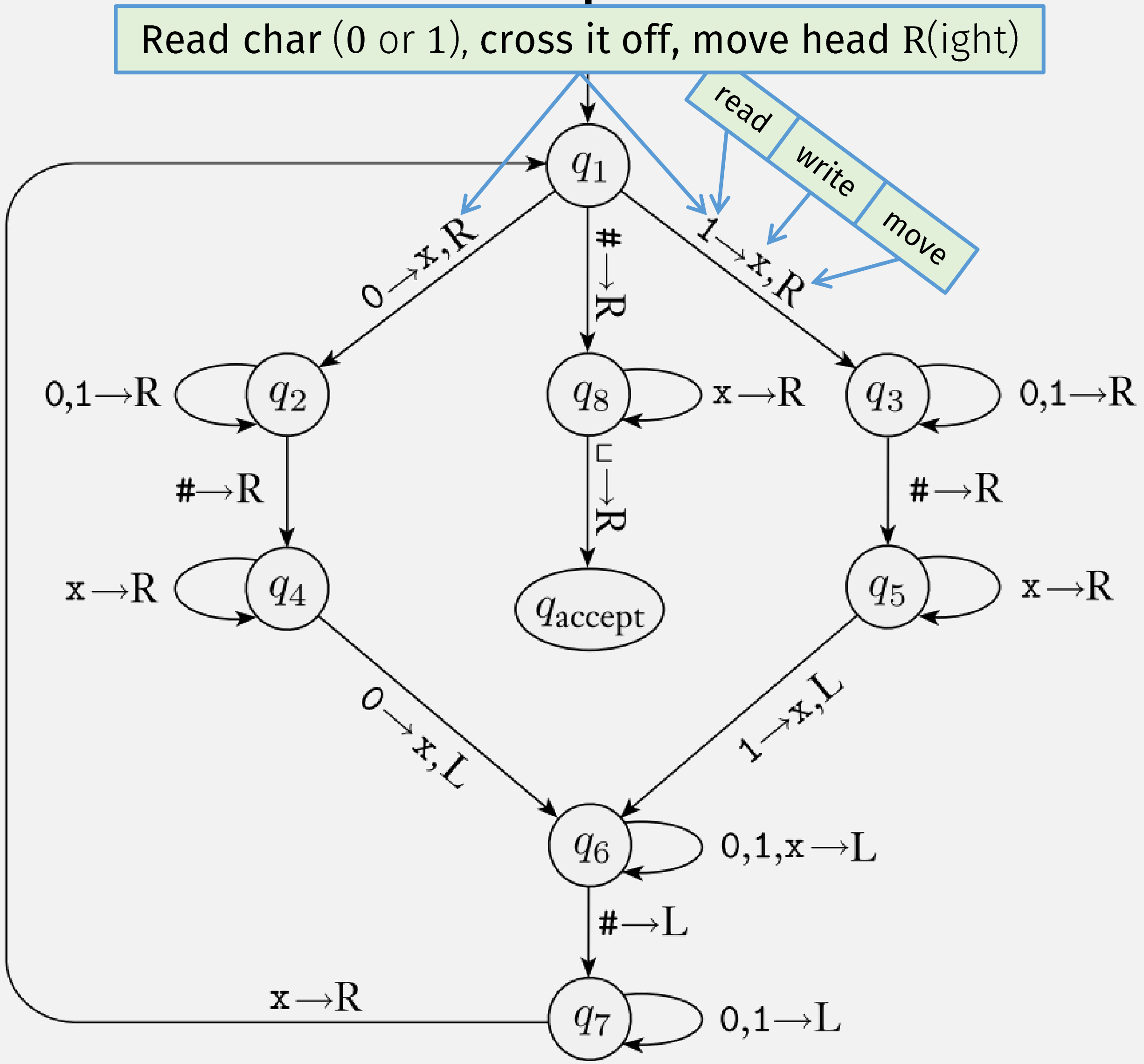
A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where $Q$, $\Sigma$, $\Gamma$ are all finite sets and

1. $Q$ is the set of states,
2. $\Sigma$ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. $\Gamma$ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta \colon Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function,
   - read: $Q \times \Gamma$
   - write: $Q \times \Gamma$
   - move: $\{\mathrm{L}, \mathrm{R}\}$
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.

Is this machine **deterministic**?
Or **non-deterministic**?

$B = \{w\#w \mid w \in \{0,1\}^*\}$

# Formal Turing Machine Example

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4. $\delta: Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,
5. $q_0 \in$ read e s write move
6. $q_{accept} \in Q$ is the accept state, and
7. $q_{reject} \in Q$ is the reject state, where $q_{reject} \neq q_{accept}$.

$B = \{w\#w \mid w \in \{0,1\}^*\}$

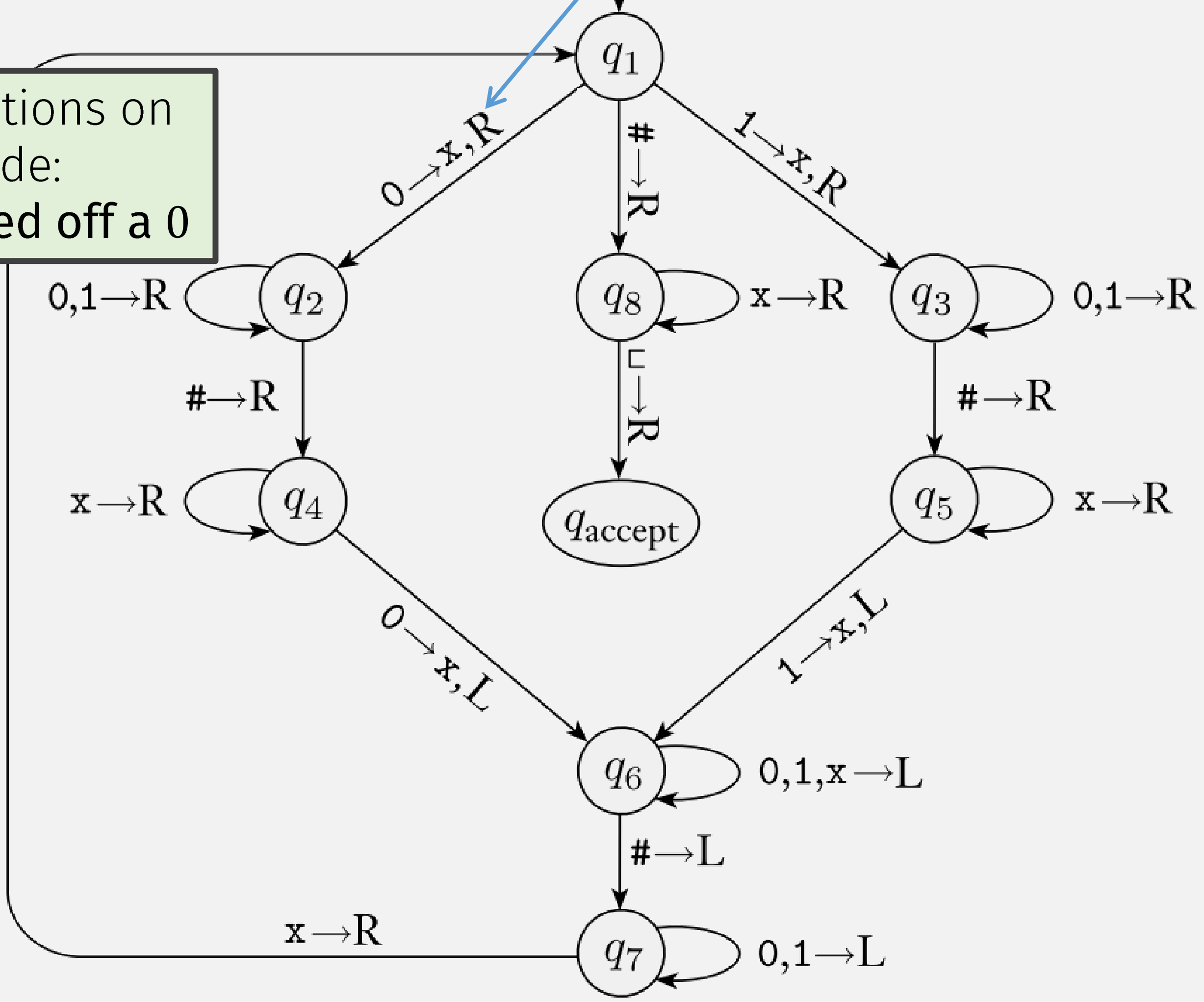
# Formal Turing Machine Example

Read char (0 ~~or 1~~), cross it off, move head R(ight)

0 1 1 0 0 0 # 0 1 1 0 0 0 ⊔ . . .

x 1 1 0 0 0 # 0 1 1 0 0 0 ⊔ . . .

Transitions on this side:
Crossed off a 0




A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where $Q$, $\Sigma$, $\Gamma$ are all finite sets and

1. $Q$ is the set of states,
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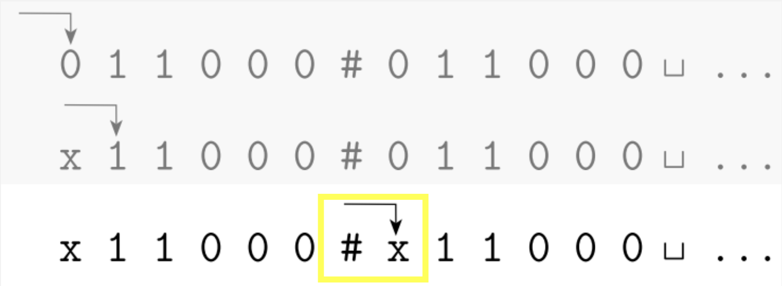
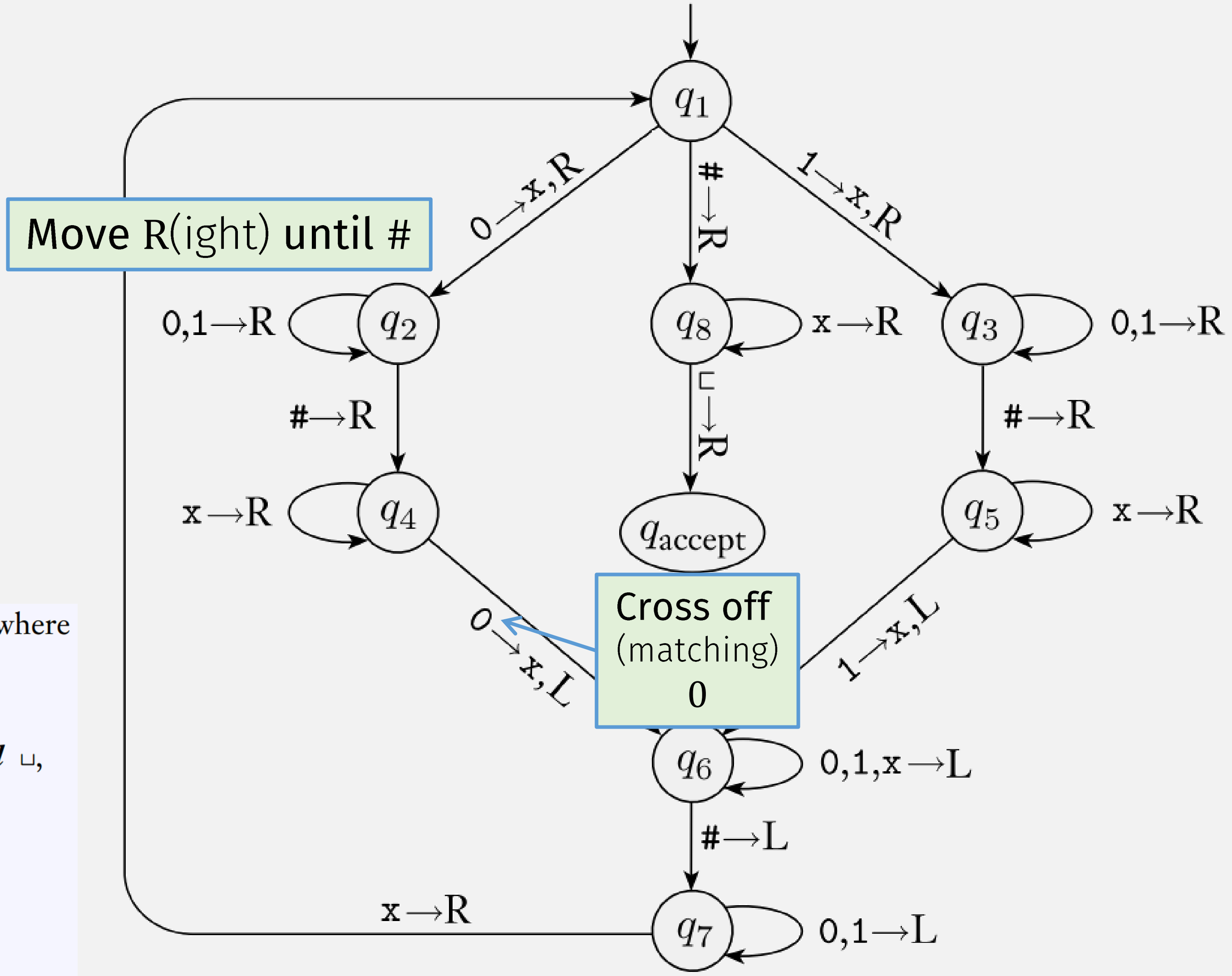
$B = \{w\#w \mid w \in \{0,1\}^*\}$

# Formal Turing Machine Example

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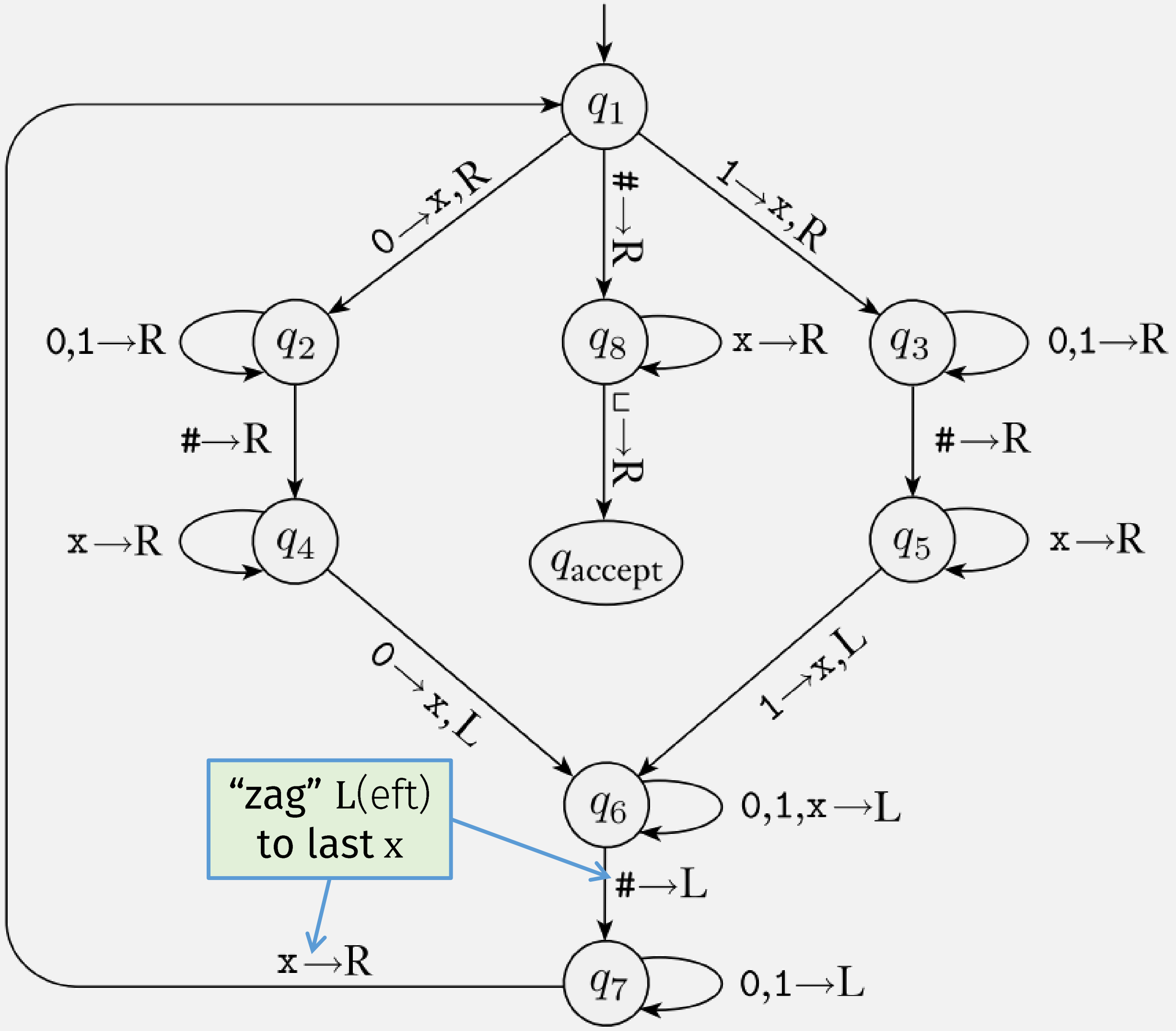
# Formal Turing Machine Example

0 1 1 0 0 0 # 0 1 1 0 0 0 ⊔ ...

x 1 1 0 0 0 # 0 1 1 0 0 0 ⊔ ...

x 1 1 0 0 0 # x 1 1 0 0 0 ⊔ ...

x 1 1 0 0 0 # x 1 1 0 0 0 ⊔ ...

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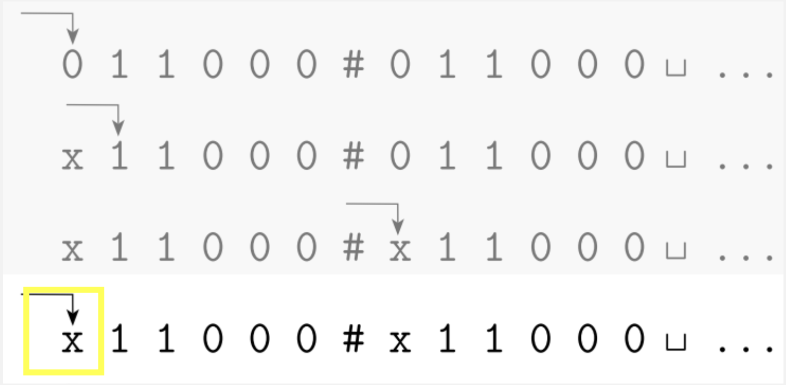


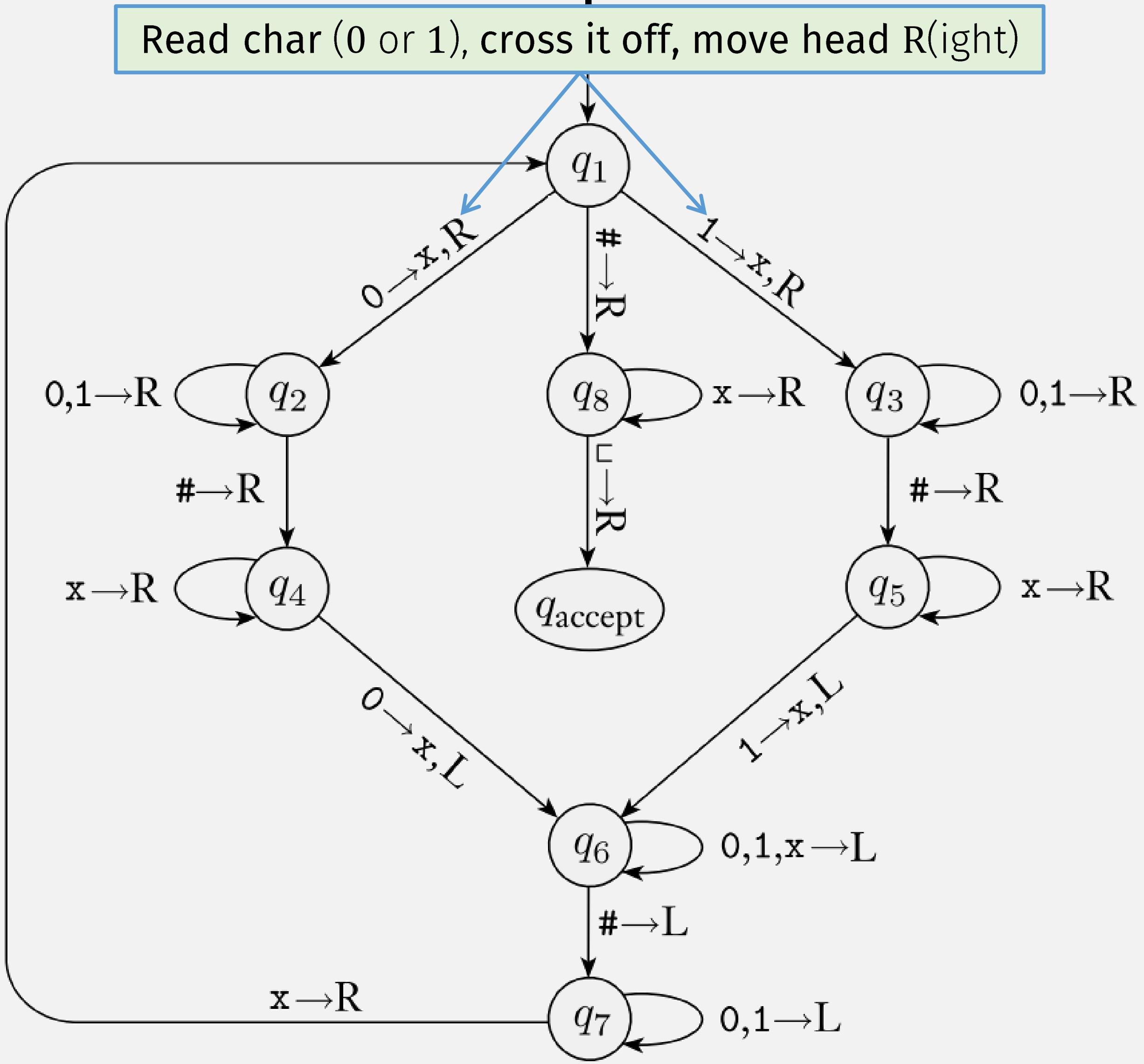


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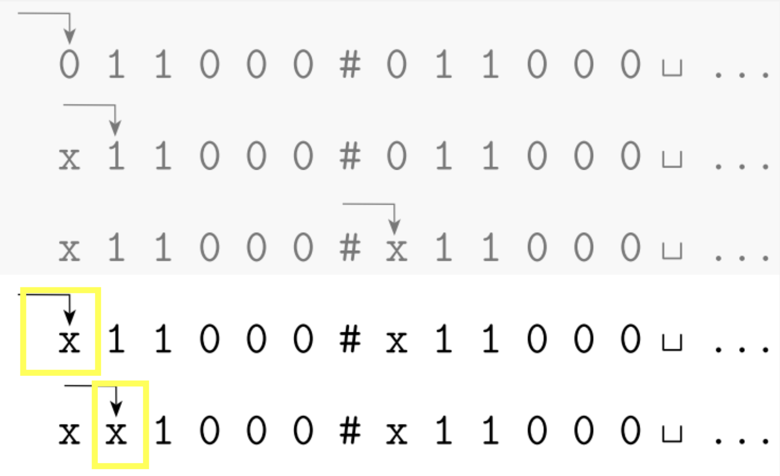
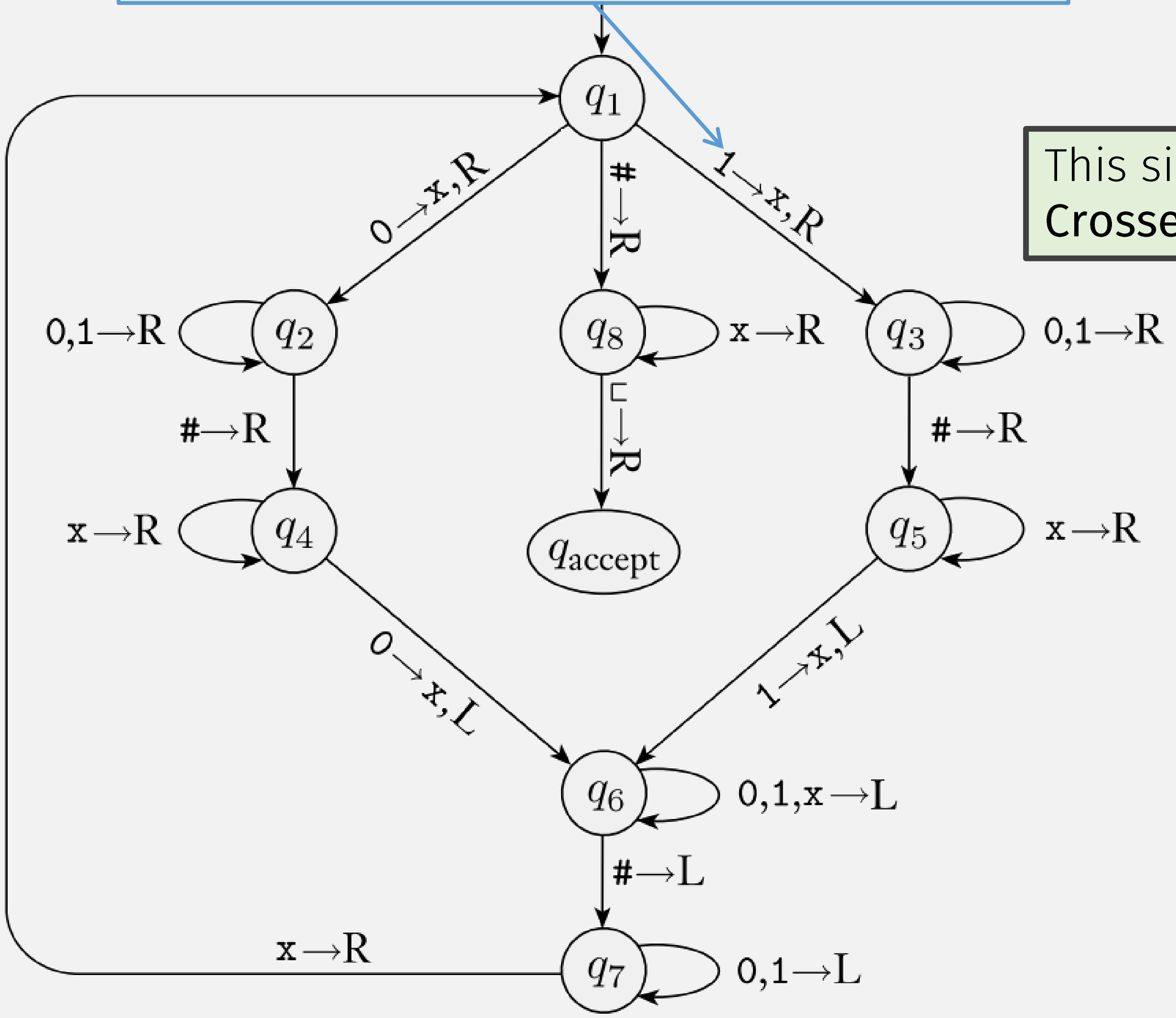
# Formal Turing Machine Example




Read char (~~0 or~~ 1), cross it off, move head R(ight)




This side:
Crossed off a 1

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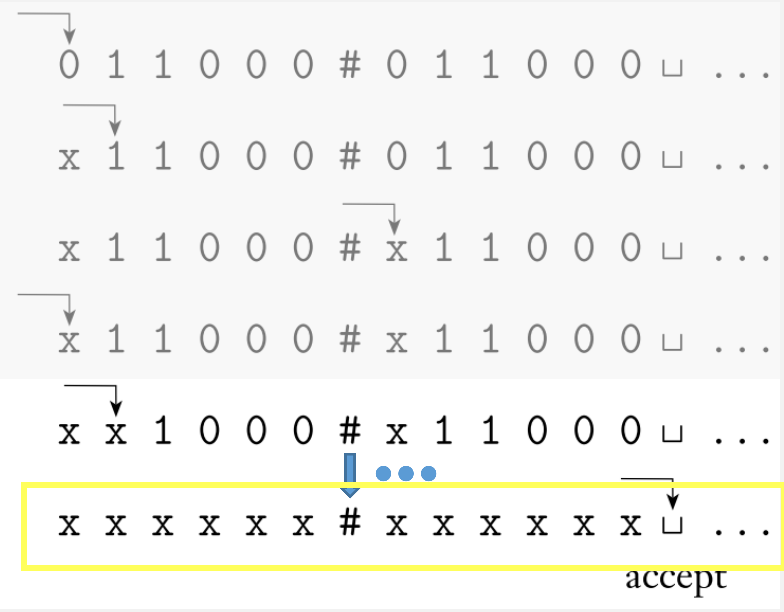
# Formal Turing Machine Example

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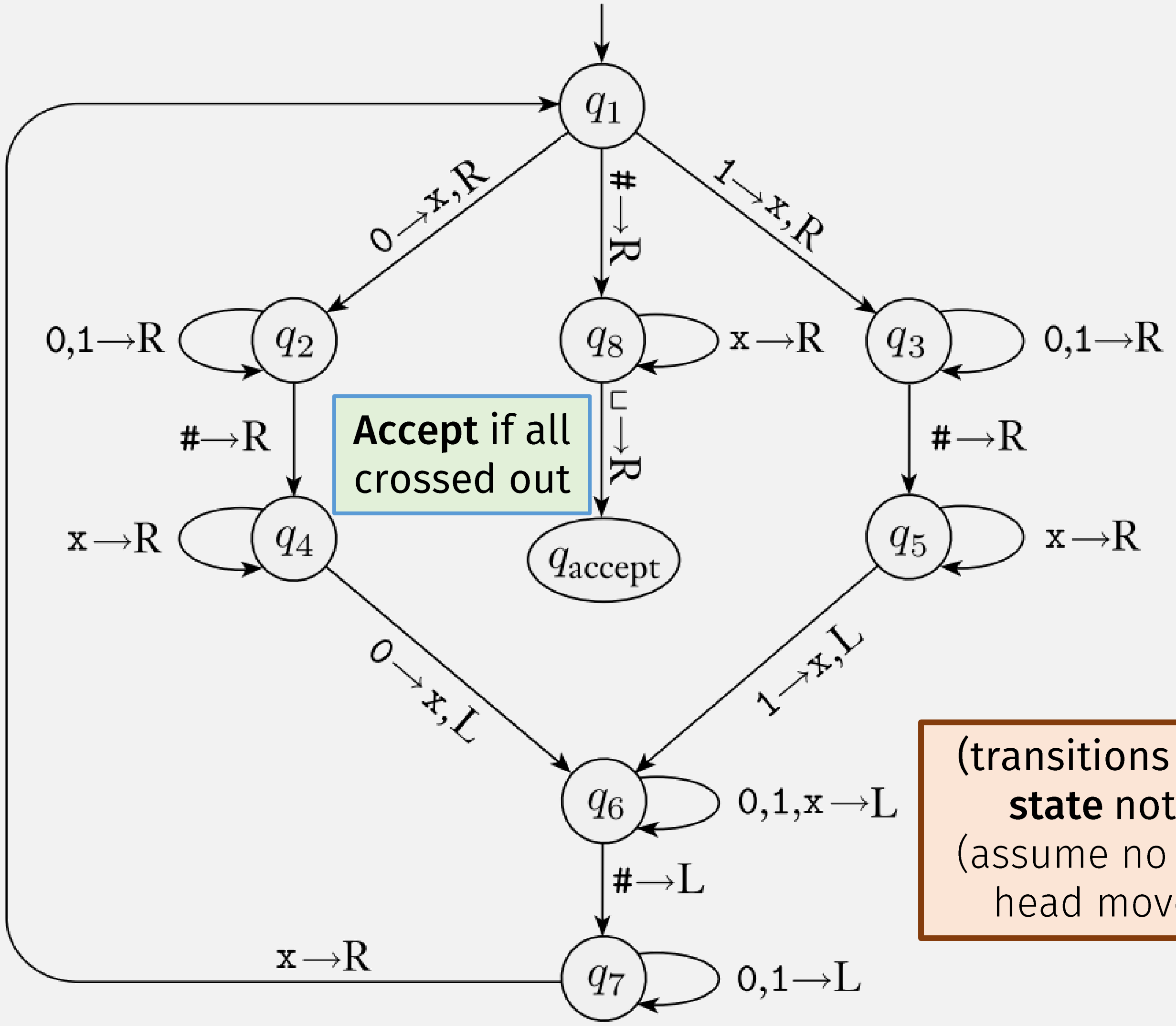



(transitions to) **Reject state not shown** (assume no write, and head moves right)

# TM Computation, Formally …

# *Flashback:* PDA Configurations (IDs)

- A **configuration** (or **ID**) is a "snapshot" of a PDA's computation

- 3 components $(q, w, \gamma)$ :

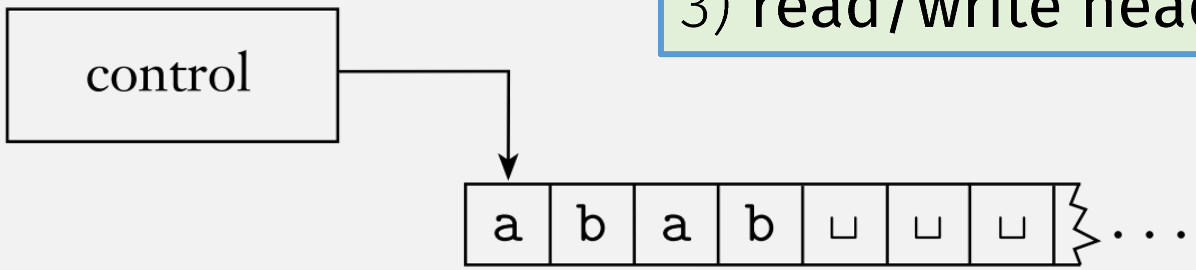
  $q$ = the current state
  $w$ = the remaining input string
  $\gamma$ = the stack contents

A **sequence of configurations** represents a **PDA** computation

# TM Configuration (ID) = ???

1) states

3) read/write head

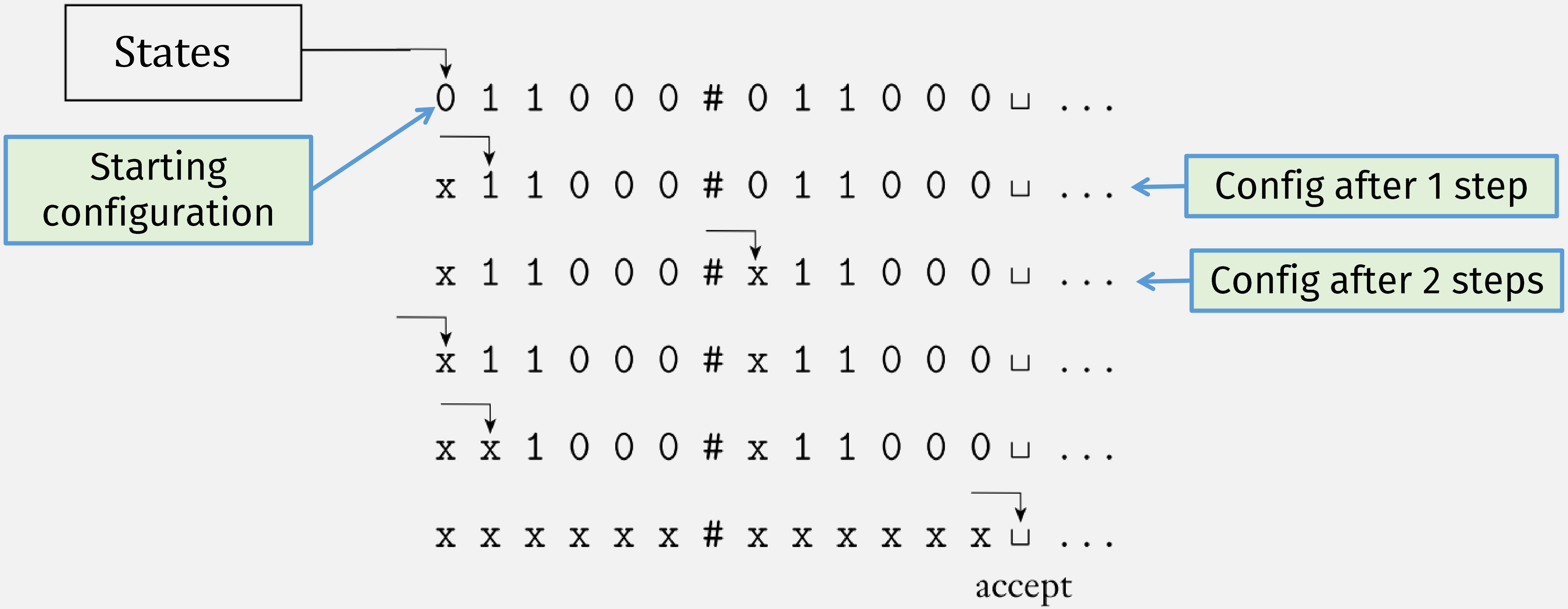



2) Tape contents

A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where $Q$, $\Sigma$, $\Gamma$ are all finite sets and

1. $Q$ is the set of states,
2. $\Sigma$ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. $\Gamma$ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{accept} \in Q$ is the accept state, and
7. $q_{reject} \in Q$ is the reject state, where $q_{reject} \neq q_{accept}$.

# TM Configuration = State + Head + Tape



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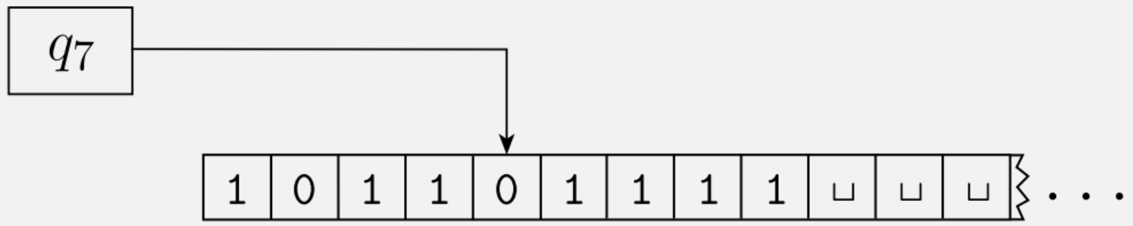



$$1011q_701111$$

Textual representation of "configuration" (use this in HW)

1st char after state is current head position

# TM Computation, Formally

Sipser says:
"For completeness, we say that the **head moves Right** in … **transitions to the reject state**"

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$$

## Single-step

(Right) $\alpha q_1 \mathbf{a} \beta \vdash \alpha \mathbf{x} q_2 \beta$

head

Next config (head moved past written char)

if $q_1, q_2 \in Q$

$\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{R})$

write

read

$\mathbf{a}, \mathbf{x} \in \Gamma \quad \alpha, \beta \in \Gamma^*$

(Left) $\alpha b q_1 \mathbf{a} \beta \vdash \alpha q_2 b \mathbf{x} \beta$

head

if $\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{L})$

(wrote $\mathbf{x}$ and) head moved left

## Multi-step

- Base Case

  $I \overset{*}{\vdash} I$ for any ID $I$

- Recursive Case

  $I \overset{*}{\vdash} J$ if there exists some ID $K$ such that $I \vdash K$ and $K \overset{*}{\vdash} J$

Edge cases:

$q_1 \mathbf{a} \beta \vdash q_2 \mathbf{x} \beta$ if $\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{L})$

Head stays at leftmost cell

(L **move,** when **already at leftmost cell**)

$\alpha q_1 \vdash \alpha \sqcup q_2$ if $\delta(q_1, \sqcup) = (q_2, \sqcup, \mathrm{R})$

Add blank symbol to config

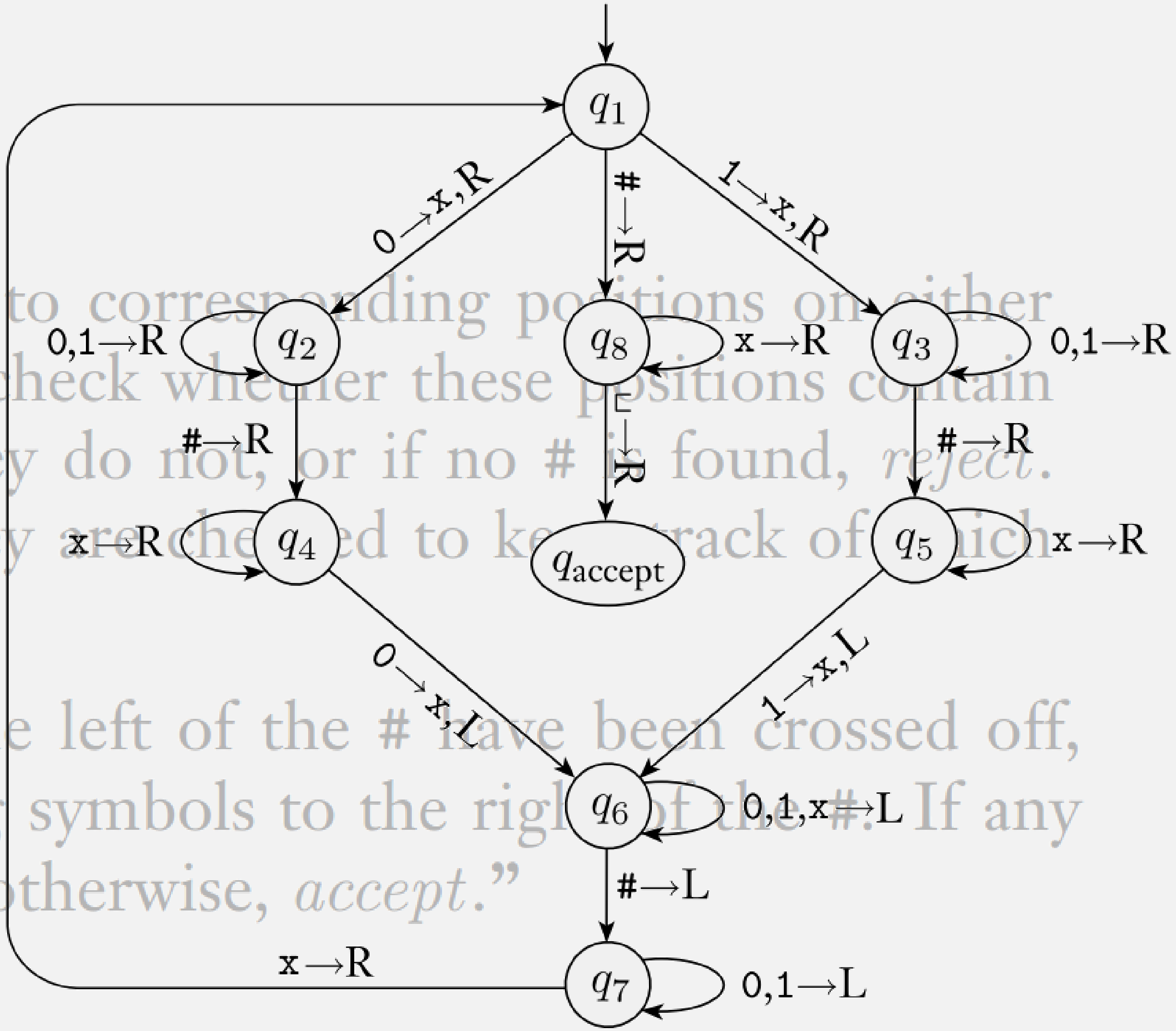
(R **move,** when **at rightmost filled cell**)

# TMs: High-level vs Low-level?

$M_1$ = "On input string $w$:

1. Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do not, or if no # is found, *reject*. Cross off symbols as they are checked to keep track of which symbols correspond.

2. When all symbols to the left of the # have been crossed off, check for any remaining symbols to the right of the #. If any symbols remain, *reject*; otherwise, *accept*."



# Turing Machine: High-level Description

- $M_1$ accepts if input is in language $B = \{w\#w \mid w \in \{0,1\}^*\}$

$M_1$ = "On input string $w$:

1. Zig-zag across the tape to corresponding positions on either side of the # symbol to check whether these positions contain the same symbol. If they do no… if no # is found, *reject*. Cross off symbols as they … track of which symbols correspond.

2. When all symbols to … n crossed off, check for any remaining … t of the #. If any symbols remain, *reject*; otherwise, … ccept."

We will (mostly) stick to **high-level** descriptions of Turing machines, like this one

# TM High-level Description Tips

Analogy:
- **High-level** TM description **~ function definition** in **"high level" language,** e.g. **Python**
- **Low-level TM tuple ~ function definition** in **bytecode** or **assembly**

**TM high-level descriptions** are <u>not</u> a **"do whatever" card,** some rules:

1. **TMs and input strings** must be <u>**named**</u> (like **function definitions**)
2. **Steps must be numbered**
3. **TMs** can **"call" or "simulate" other TMs** (if they pass appropriate arguments!)
   - e.g., **step for a TM** $M$ can **say: "call TM** $M_2$ **with argument string** $w$**, if** $M_2$ **accepts** $w$ **then ..., else ..."**
   - Can **split input into substrings** and **pass to different TMs**
4. **Follow typical programming "scoping" rules**
   - can **assume functions already defined** are in **"global" scope,** "CONVERT" ...
5. **Other variables must also be defined before use**
   - e.g., can **define a TM inside another TM**
6. must be **equivalent** to a low-level formal tuple
   - high-level "<u>step</u>" represents a <u>finite</u> # of low-level δ transitions
   - So one step cannot run forever
   - E.g., can't **say "try all numbers" as a "step"**

$M_1 =$ "On input string $w$:

$M =$ "On input $w$
1. Simulate $B$ on input $w$.
2. If simulation ends in accept state,

$N =$ "On input $\langle B, w\rangle$, where $B$ is an NFA and $w$ is a string:
1. Convert NFA $B$ to an equivalent DFA $C$, using the procedur
this conversion given in Theorem 1.39.
2. Run TM $M$ from Theorem 4.1 on input $\langle C, w\rangle$.

$S =$ "On input $w$
1. Construct the following TM $M_2$.
$M_2 =$ "On input $x$:

# Non-halting Turing Machines (TMs)

IMPORTANT SLIDE

So: **TM computation** has
<u>**3 possible results**</u>:
- **Accept**
- **Reject**
- **Loop forever**

- **A Turing Machine** can <u>**run forever**</u>
  - E.g., **head** can **move back and forth in a loop**

- We will work with <u>**two classes of Turing Machines**</u>:
  - A **recognizer** is a **Turing Machine** that **may run forever** (all possible TMs)
  - A **decider** is a **Turing Machine** that **always halts.**

Call a language ***Turing-recognizable*** if some Turing machine recognizes it.

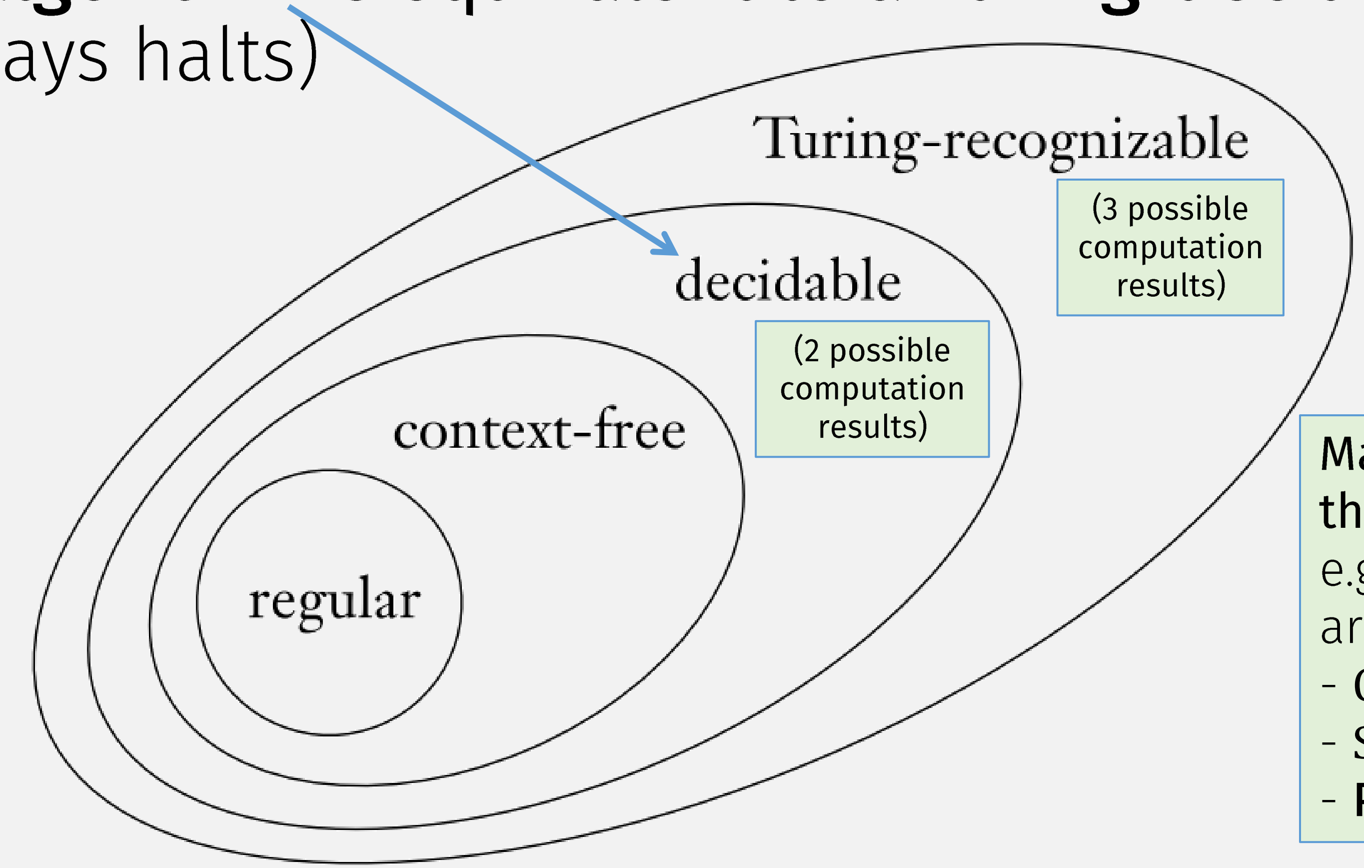
(3 possible computation results)

Call a language ***Turing-decidable*** or simply ***decidable*** if some Turing machine decides it.

(2 possible computation results)

# Formal Definition of an "Algorithm"

- An **algorithm** is equivalent to a **Turing-decidable** Language (always halts)




Many <u>functions</u> we have **defined this semester** are **algorithms!**
e.g., **all our** <u>conversion</u> **functions** are **decider** TMs!!
- $\text{CONVERT}_{\text{DFA-NFA}}$
- $\text{STAR}_{\text{NFA}}$
- **PDA→CFG**