Seminar in Differential Geometry and General Relativity
Suggested Problem Set 1 (Draft)
5 June 2007

(1) Look up the axioms that define a vector space. Then show that if V is a vector space and $v \in V$ then $(-1) \cdot v = -v$ using on the axioms. There is actually something to prove here. On the left-hand side of the equation, we have a scalar multiplied by a vector. On the right-hand, there's only a vector. The minus sign is part of the symbol $-v$ that means "the additive inverse of a vector named v."

(2) Recall the vector space $C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$. For each $c \in \mathbb{R}$, consider the set of vectors defined by

$$V_c = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous and } f(0) = c\}.$$

For which values of c is V_c a subspace of $C[0,1]$? Explain.

(3) Consider the map $e_p : C[0,1] \to \mathbb{R}$ that evaluates a function f at p, that is, $e_p(f) = f(p)$. For which values of p, if any, is the evaluation map e_p linear?

(4) Find an infinite set of linearly independent vectors in $C[0,1]$, not including polynomials. If you can, prove that your set is indeed linearly independent. Short of that, explain what you need to do in order to prove such a claim.

(5) Show that any two vectors $v = (v_1, v_2)$, $w = (w_1, w_2)$ in $\mathbb{R}^2$ form a basis if and only if the components of those vectors satisfy $v_1 w_2 - v_2 w_1 \neq 0$. How is this statement related to invertibility of a change of basis operator?

(6) Let P_3 denote the vector space consisting of polynomials of degree less than or equal to 3, with real coefficients. Let $D : P_3 \to P_3$ denote the usual derivitive operator, and let $\mathrm{Id} : P_3 \to P_3$ be the identity operator; that is $\mathrm{Id} : p \mapsto p$ for all $p \in P_3$.

Now consider the following three bases of P_3: $\mathcal{B}_1 = \{1, x, x^2, x^3\}$, $\mathcal{B}_2 = \{1, 1+x, 1+x^2, 1+x^3\}$, $\mathcal{B}_3 = \{1+x, 1-x, x^2 - x^3, x^2 + x^3\}$.

(a) Convince yourself that P_3 is a vector space and that $\mathcal{B}_i$ are each a basis for it.
(b) Write down the matrix of D with respect to each of the three bases.
(c) Write down the matrix for Id where the domain and range both have $\mathcal{B}_3$ as a basis.
(d) Write down the matrix for Id where the domain has basis $\mathcal{B}_1$ and the range has basis $\mathcal{B}_2$.