

Random Variables

In some experiments, we would like to assign a numerical value to each possible outcome in order to facilitate a mathematical analysis of the experiment.

For this purpose, we introduce **random variables**.

Definition: A random variable is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

Note: Random variables are functions, not variables, and they are not random, but map random results from experiments onto real numbers in a well-defined manner.

March 27, 2018

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1

Random Variables

Example:

Let X be the result of a rock-paper-scissors game. If player A chooses symbol a and player B chooses symbol b , then

$$\begin{aligned} X(a, b) &= 1, \text{ if player A wins,} \\ &= 0, \text{ if A and B choose the same symbol,} \\ &= -1, \text{ if player B wins.} \end{aligned}$$

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2

Random Variables

$X(\text{rock, rock}) =$	0
$X(\text{rock, paper}) =$	-1
$X(\text{rock, scissors}) =$	1
$X(\text{paper, rock}) =$	1
$X(\text{paper, paper}) =$	0
$X(\text{paper, scissors}) =$	-1
$X(\text{scissors, rock}) =$	-1
$X(\text{scissors, paper}) =$	1
$X(\text{scissors, scissors}) =$	0

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3

Expected Values

Once we have defined a random variable for our experiment, we can statistically analyze the outcomes of the experiment.

For example, we can ask: What is the **average value** (called the **expected value**) of a random variable when the experiment is carried out a large number of times?

Can we just calculate the arithmetic mean across all possible values of the random variable?

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4

Expected Values

No, we cannot, since it is possible that some outcomes are more likely than others.

For example, assume the possible outcomes of an experiment are 1 and 2 with probabilities of 0.1 and 0.9, respectively.

Is the average value 1.5?

No, since 2 is much more likely to occur than 1, the average must be larger than 1.5.

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5

Expected Values

Instead, we have to calculate the **weighted sum** of all possible outcomes, that is, each value of the random variable has to be multiplied with its probability before being added to the sum.

In our example, the average value is given by $0.1 \cdot 1 + 0.9 \cdot 2 = 0.1 + 1.8 = 1.9$.

Definition: The **expected value** (or expectation) of the random variable $X(s)$ on the sample space S is equal to:

$$E(X) = \sum_{s \in S} p(s)X(s).$$

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6

Expected Values

Example: Let X be the random variable equal to the sum of the numbers that appear when a pair of dice is rolled.

There are **36 outcomes** (= pairs of numbers from 1 to 6).

The **range** of X is $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Are the 36 outcomes equally likely?

Yes, if the dice are not biased.

Are the 11 values of X equally likely to occur?

No, the probabilities vary across values.

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7

Expected Values

$$P(X = 2) = 1/36$$

$$P(X = 3) = 2/36 = 1/18$$

$$P(X = 4) = 3/36 = 1/12$$

$$P(X = 5) = 4/36 = 1/9$$

$$P(X = 6) = 5/36$$

$$P(X = 7) = 6/36 = 1/6$$

$$P(X = 8) = 5/36$$

$$P(X = 9) = 4/36 = 1/9$$

$$P(X = 10) = 3/36 = 1/12$$

$$P(X = 11) = 2/36 = 1/18$$

$$P(X = 12) = 1/36$$

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8

Expected Values

$$E(X) = 2 \cdot (1/36) + 3 \cdot (1/18) + 4 \cdot (1/12) + 5 \cdot (1/9) + 6 \cdot (5/36) + 7 \cdot (1/6) + 8 \cdot (5/36) + 9 \cdot (1/9) + 10 \cdot (1/12) + 11 \cdot (1/18) + 12 \cdot (1/36)$$

$$E(X) = 7$$

This means that if we roll the dice many times, sum all the numbers that appear and divide the sum by the number of trials, we expect to find a value of 7.

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9

Expected Values

Theorem:

If X and Y are random variables on a sample space S , then $E(X + Y) = E(X) + E(Y)$.

Furthermore, if X_i , $i = 1, 2, \dots, n$ with a positive integer n , are random variables on S , then $E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$.

Moreover, if a and b are real numbers, then $E(aX + b) = aE(X) + b$.

Note the typo in the textbook!

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10

Expected Values

Knowing this theorem, we could now solve the previous example much more easily:

Let X_1 and X_2 be the numbers appearing on the first and the second die, respectively.

For each die, there is an equal probability for each of the six numbers to appear. Therefore, $E(X_1) = E(X_2) = (1 + 2 + 3 + 4 + 5 + 6)/6 = 7/2$.

We now know that

$$E(X_1 + X_2) = E(X_1) + E(X_2) = 7.$$

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11

Expected Values

We can use our knowledge about expected values to compute the average-case complexity of an algorithm.

Let the sample space be the set of all possible inputs a_1, a_2, \dots, a_n , and the random variable X assign to each a_j the number of operations that the algorithm executes for that input.

For each input a_j , the probability that this input occurs is given by $p(a_j)$.

The algorithm's average-case complexity then is:

$$E(X) = \sum_{j=1, \dots, n} p(a_j) X(a_j)$$

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12

Expected Values

However, in order to conduct such an average-case analysis, you would need to find out:

- the number of steps that the algorithms takes for any (!) possible input, and
- the probability for each of these inputs to occur.

For most algorithms, this would be a highly complex task, so we will stick with the worst-case analysis.

In the textbook (page 283 in the 4th Edition, page 384 in the 5th Edition, page 431 in the 6th Edition, page 482 in the 7th Edition), an average-case analysis of the linear search algorithm is shown.

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13

Independent Random Variables

Definition: The random variables X and Y on a sample space S are **independent** if

$$p(X(s) = r_1 \wedge Y(s) = r_2) = p(X(s) = r_1) \cdot p(Y(s) = r_2).$$

In other words, X and Y are independent if the probability that $X(s) = r_1 \wedge Y(s) = r_2$ equals the product of the probability that $X(s) = r_1$ and the probability that $Y(s) = r_2$ for all real numbers r_1 and r_2 .

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14

Independent Random Variables

Example: Are the random variables X_1 and X_2 from the "pair of dice" example independent?

Solution:

$$p(X_1 = i) = 1/6$$

$$p(X_2 = j) = 1/6$$

$$p(X_1 = i \wedge X_2 = j) = 1/36$$

Since $p(X_1 = i \wedge X_2 = j) = p(X_1 = i) \cdot p(X_2 = j)$, the random variables X_1 and X_2 are **independent**.

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15

Independent Random Variables

Theorem: If X and Y are independent random variables on a sample space S , then $E(XY) = E(X)E(Y)$.

Note:

$E(X + Y) = E(X) + E(Y)$ is true for any X and Y , but $E(XY) = E(X)E(Y)$ needs X and Y to be independent.

How come?

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16

Independent Random Variables

Example: Let X and Y be random variables on some sample space, and each of them assumes the values 1 and 3 with equal probability.

Then $E(X) = E(Y) = 2$

If X and Y are **independent**, we have:

$$E(X + Y) = 1/4 \cdot (1 + 1) + 1/4 \cdot (1 + 3) + 1/4 \cdot (3 + 1) + 1/4 \cdot (3 + 3) = 4 = E(X) + E(Y)$$

$$E(XY) = 1/4 \cdot (1 \cdot 1) + 1/4 \cdot (1 \cdot 3) + 1/4 \cdot (3 \cdot 1) + 1/4 \cdot (3 \cdot 3) = 4 = E(X) \cdot E(Y)$$

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17

Independent Random Variables

Let us now assume that X and Y are **correlated** in such a way that $Y = 1$ whenever $X = 1$, and $Y = 3$ whenever $X = 3$.

$$E(X + Y) = 1/2 \cdot (1 + 1) + 1/2 \cdot (3 + 3) = 4 = E(X) + E(Y)$$

$$E(XY) = 1/2 \cdot (1 \cdot 1) + 1/2 \cdot (3 \cdot 3) = 5 \neq E(X) \cdot E(Y)$$

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18

Variance

The **expected value** of a random variable is an important parameter for the description of a random distribution.

It does not tell us, however, anything about **how widely distributed** the values are.

This can be described by the **variance** of a random distribution.

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19

Variance

Definition: Let X be a random variable on a sample space S . The **variance** of X , denoted by $V(X)$, is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

The **standard deviation** of X , denoted by $\sigma(X)$, is defined to be the square root of $V(X)$.

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20

Variance

Useful rules:

If X is a random variable on a sample space S , then $V(X) = E(X^2) - E(X)^2$.

If X and Y are two independent random variables on a sample space S , then $V(X + Y) = V(X) + V(Y)$.

Furthermore, if X_i , $i = 1, 2, \dots, n$, with a positive integer n , are pairwise independent random variables on S , then

$$V(X_1 + X_2 + \dots + X_n) = V(X_1) + V(X_2) + \dots + V(X_n).$$

Proofs in the textbook on pages 280 and 281 (4th Edition), pages 389 and 390 (5th Edition), pages 436 and 437 (6th Edition), and pages 488-490 (7th Edition).

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21

Now it's Time for...

Advanced Counting Techniques

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22

Recurrence Relations

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

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23

Recurrence Relations

In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely determined**.

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24

Recurrence Relations

Example:

Consider the recurrence relation
 $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$

Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

For $n \geq 2$ we see that

$$2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n.$$

Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

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25

Recurrence Relations

Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?

For $n \geq 2$ we see that

$$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n.$$

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

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26

Modeling with Recurrence Relations

Example:

Someone deposits \$10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years.

How can we determine P_n on the basis of P_{n-1} ?

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Week 8: Advanced Counting

27

Modeling with Recurrence Relations

We can derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}.$$

The initial condition is $P_0 = 10,000$.

Then we have:

$$P_1 = 1.05P_0$$

$$P_2 = 1.05P_1 = (1.05)^2P_0$$

$$P_3 = 1.05P_2 = (1.05)^3P_0$$

...

$$P_n = 1.05P_{n-1} = (1.05)^nP_0$$

We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

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28

Modeling with Recurrence Relations

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$$P_{30} = (1.05)^{30} \cdot 10,000 = 43,219.42$$

After 30 years, the account contains \$43,219.42.

March 27, 2018

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29