Combining Relations

We already know that **functions** are just **special cases** of **relations** (namely those that map each element in the domain onto exactly one element in the codomain).

If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).

Combining Relations

Definition: Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, …, are defined inductively by

$R^1 = R$

$R^{n+1} = R^n \circ R$

In other words:

$R^n = R \circ R \circ \ldots \circ R$ (n times the letter R)

Combining Relations

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

Remember the definition of transitivity:

Definition: A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

The composite of R with itself contains exactly these pairs (a, c).

Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.

Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.

Representing Relations

We already know different ways of representing relations. We will now take a closer look at two ways of representation: **Zero-one matrices** and **directed graphs**.

If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

$m_{ij} = 1$, if $(a_i, b_j) \in R$, and

$m_{ij} = 0$, if $(a_i, b_j) \notin R$.

Note that for creating this matrix we first need to list the elements in A and B in a **particular, but arbitrary order**.

Representing Relations

Example: How can we represent the relation R from the set A = {1, 2, 3} to the set B = {1, 2} with R = {(2, 1), (3, 1), (3, 2)} as a zero-one matrix?

Solution: The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations

What do we know about the matrices representing a **relation on a set** (a relation from A to A) ?

They are **square** matrices.

What do we know about matrices representing **reflexive** relations?

All the elements on the **diagonal** of such matrices M_{ref} must be **1s**.

$$M_{ref} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

Representing Relations

What do we know about the matrices representing **symmetric relations**?

These matrices are symmetric, that is, $M_R = (M_R)^t$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

symmetric matrix, symmetric relation. non-symmetric matrix, non-symmetric relation.

Representing Relations

The Boolean operations **join** and **meet** (you remember?) can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

To obtain the **join** of two zero-one matrices, we apply the Boolean "or" function to all corresponding elements in the matrices.

To obtain the **meet** of two zero-one matrices, we apply the Boolean "and" function to all corresponding elements in the matrices.

Representing Relations

Example: Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R \cup S$ and $R \cap S$?

Solution: These matrices are given by

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Matrices

Do you remember the **Boolean product** of two zero-one matrices?

Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix.

Then the **Boolean product** of A and B, denoted by $A \circ B$, is the $m \times n$ matrix with (i, j)th entry $[c_{ij}]$, where

$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$.

$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

Representing Relations Using Matrices

Let us now assume that the zero-one matrices $M_A = [a_{ij}]$, $M_B = [b_{ij}]$ and $M_C = [c_{ij}]$ represent relations A, B, and C, respectively.

Remember: For $M_C = M_A \circ M_B$ we have:

$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

In terms of the **relations**, this means that C contains a pair (x_i, z_j) if and only if there is an element y_n such that (x_i, y_n) is in relation A and (y_n, z_j) is in relation B.

Therefore, $C = B \circ A$ (**composite** of A and B).

Representing Relations Using Matrices

This gives us the following rule:

$M_{B \circ A} = M_A \circ M_B$

In other words, the matrix representing the **composite** of relations A and B is the **Boolean product** of the matrices representing A and B.

Analogously, we can find matrices representing the **powers of relations**:

$M_{R^n} = M_R^{[n]}$ (n-th **Boolean power**).

Representing Relations Using Matrices

Example: Find the matrix representing R^2, where the matrix representing R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is given by

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

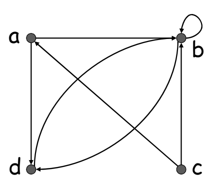
Definition: A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).

The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

We can use arrows to display graphs.

Representing Relations Using Digraphs

Example: Display the digraph with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop.**

Representing Relations Using Digraphs

Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs (a, b)$\in$R as its edges.

Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.

This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

Closures of Relations

What is the **closure** of a relation?

Definition: Let R be a relation on a set A. R may or may not have some **property P**, such as reflexivity, symmetry, or transitivity.

If there is a relation S that contains R and has property P, and S is a subset of **every** relation that contains R and has property P, then S is called the **closure** of R with respect to P.

Note that the closure of a relation with respect to a property may not exist.

Closures of Relations

Example I: Find the **reflexive closure** of relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3}.

Solution: We know that any reflexive relation on A must contain the elements (1, 1), (2, 2), and (3, 3).

By adding (2, 2) and (3, 3) to R, we obtain the reflexive relation S, which is given by
S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3)}.

S is reflexive, contains R, and is contained within every reflexive relation that contains R.

Therefore, S is the **reflexive closure** of R.