Graph Theorems

Theorem: An undirected graph has an even number of vertices of odd degree.

Idea: There are three possibilities for adding an edge to connect two vertices in the graph:

Before:		**After:**
Both vertices have even degree	⇨	Both vertices have odd degree
Both vertices have odd degree	⇨	Both vertices have even degree
One vertex has odd degree, the other even	⇨	One vertex has even degree, the other odd

Graph Theorems

There are two possibilities for adding a loop to a vertex in the graph:

Before:		**After:**
The vertex has even degree	⇨	The vertex has even degree
The vertex has odd degree	⇨	The vertex has odd degree

Graph Terminology

So if there is an even number of vertices of odd degree in the graph, it will still be even after adding an edge.

Therefore, since an undirected graph with **no edges** has an even number of vertices with odd degree (zero), the same must be true for **any** undirected graph.

Please also study the proof on

4^{th} Edition: page 446

5^{th} Edition: page 547

6^{th} edition: page 599

7^{th} edition: page 653

Graph Terminology

Definition: When (u, v) is an edge of the graph G with directed edges, u is said to be **adjacent to** v, and v is said to be **adjacent from** u.

The vertex u is called the **initial vertex** of (u, v), and v is called the **terminal vertex** of (u, v).

The initial vertex and terminal vertex of a loop are the same.

Graph Terminology

Definition: In a graph with directed edges, the **in-degree** of a vertex v, denoted by **$\deg^-(v)$**, is the number of edges with v as their **terminal vertex**.

The **out-degree** of v, denoted by **$\deg^+(v)$**, is the number of edges with v as their initial vertex.

Question: How does adding a loop to a vertex change the in-degree and out-degree of that vertex?

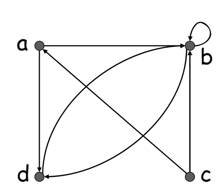
Answer: It increases both the in-degree and the out-degree by one.

Graph Terminology

Example: What are the in-degrees and out-degrees of the vertices a, b, c, d in this graph:

$\deg^-(a) = 1$
$\deg^+(a) = 2$

$\deg^-(b) = 4$
$\deg^+(b) = 2$



$\deg^-(d) = 2$
$\deg^+(d) = 1$

$\deg^-(c) = 0$
$\deg^+(c) = 2$

Graph Terminology

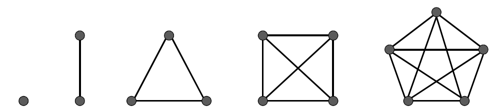
Theorem: Let G = (V, E) be a graph with directed edges. Then:

$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$

This is easy to see, because every new edge increases both the sum of in-degrees and the sum of out-degrees by one.

Special Graphs

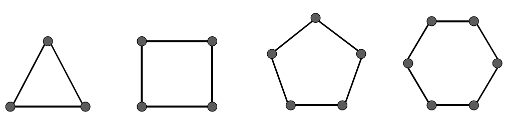
Definition: The **complete graph** on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



K_1 K_2 K_3 K_4 K_5

Special Graphs

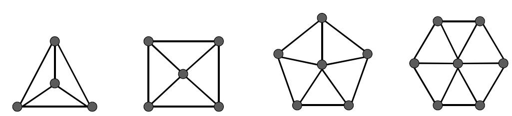
Definition: The **cycle** C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



C_3 C_4 C_5 C_6

Special Graphs

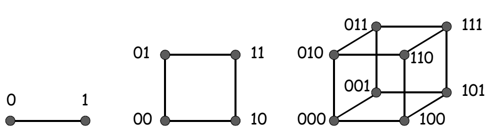
Definition: We obtain the **wheel** W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n by adding new edges.



W_3 W_4 W_5 W_6

Special Graphs

Definition: The **n-cube,** denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



Q_1 Q_2 Q_3

Special Graphs

Definition: A simple graph is called **bipartite** if its vertex set V can be partitioned into two disjoint nonempty sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 with a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

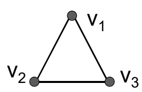
For example, consider a graph that represents each person in a mixed-doubles tennis tournament (i.e., teams consist of one female and one male player).

Players of the same team are connected by edges.

This graph is **bipartite**, because each edge connects a vertex in the **subset of males** with a vertex in the **subset of females**.

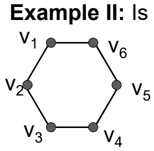
Special Graphs

Example I: Is C_3 bipartite?

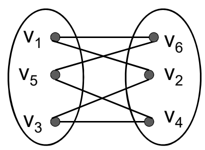


No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.

Example II: Is C_6 bipartite?

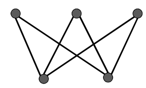


Yes, because we can display C_6 like this:

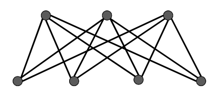


Special Graphs

Definition: The **complete bipartite graph** $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. Two vertices are connected if and only if they are in different subsets.



$K_{3,2}$



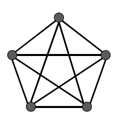
$K_{3,4}$

Operations on Graphs

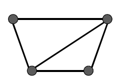
Definition: A **subgraph** of a graph G = (V, E) is a graph H = (W, F) where $W \subseteq V$ and $F \subseteq E$.

Note: Of course, H is a valid graph, so we cannot remove any endpoints of remaining edges when creating H.

Example:



K_5

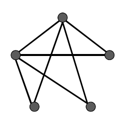


subgraph of K_5

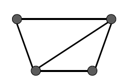
Operations on Graphs

Definition: The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

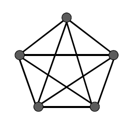
The union of G_1 and G_2 is denoted by $\mathbf{G_1 \cup G_2}$.



G_1

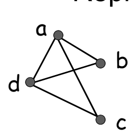


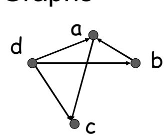
G_2



$G_1 \cup G_2 = K_5$

Representing Graphs





Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

Initial Vertex	Terminal Vertices
a	c
b	a
c	
d	a, b, c

Representing Graphs

Definition: Let G = (V, E) be a simple graph with |V| = n. Suppose that the vertices of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$.

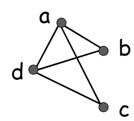
The **adjacency matrix** A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 otherwise.

In other words, for an adjacency matrix $A = [a_{ij}]$,

$a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,
$a_{ij} = 0$ otherwise.

Representing Graphs

Example: What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?



Solution: $$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Note: Adjacency matrices of undirected graphs are always symmetric.

Representing Graphs

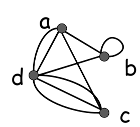
For the representation of graphs with **multiple edges**, we can no longer use zero-one matrices.

Instead, we use **matrices of natural numbers**.

The (i, j)th entry of such a matrix equals the **number of edges** that are associated with $\{v_i, v_j\}$.

Representing Graphs

Example: What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?



Solution: $$A_G = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

Note: For undirected graphs, adjacency matrices are symmetric.

Representing Graphs

Definition: Let G = (V, E) be a **directed graph** with |V| = n. Suppose that the vertices of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$.

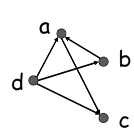
The **adjacency matrix** A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when there is an edge from v_i to v_j, and 0 otherwise.

In other words, for an adjacency matrix $A = [a_{ij}]$,

$a_{ij} = 1$ if (v_i, v_j) is an edge of G,
$a_{ij} = 0$ otherwise.

Representing Graphs

Example: What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?



Solution: $$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$