CS 320L – Applied Discrete Mathematics – Spring 2018
Instructor: Marc Pomplun

Assignment #3

Sample Solutions

Note: This is a fillable PDF, and you need to put all of your answers **into the answer boxes**. Press CTRL + E to add formatting, and you can copy and paste symbols from this or other documents (like these: $\forall$ $\exists$ $\rightarrow$ $\leftrightarrow$ $\leq$ $\geq$ $\times$ $\cdot$ $\neq$ $\cap$ $\cup$ $\subset$ $\subseteq$ $\in$ $\notin$ $\neg$). You can do this digitally, without any paper involved, or you can print this assignment, fill in the answers with a pen, and then scan the result. The copier in the department can scan and email you the scanned pages for free. Alternatively, there are scanning apps for all types of smart phones. Afterwards, upload your answers to gradescope.com. If you have not signed up yet, please do so using code 9D5E5J. **This time, hard copies will no longer be accepted.**

Question 1: Rolling Dice

Let *X* be the random variable that is defined as the smaller of the two numbers that appear when a pair of dice is rolled. For example, if you roll 2 and 5, then *X* = 2.

a) Determine the expected value and the standard deviation of *X* for two fair dice whose results are independent from each other.

X has the following equally probable 36 values:

1	1	1	1	1	1
1	2	2	2	2	2
1	2	3	3	3	3
1	2	3	4	4	4
1	2	3	4	5	5
1	2	3	4	5	6

$E(X) = 11/36\cdot1 + 9/36\cdot2 + 7/36\cdot3 + 5/36\cdot4 + 3/36\cdot5 + 1/36\cdot6 = 91/36 \approx 2.528$

$V(X) = 11/36\cdot(1 - 91/36)^2 + 9/36\cdot(2 - 91/36)^2 + 7/36\cdot(3 - 91/36)^2 + 5/36\cdot(4 - 91/36)^2$
$\quad\quad + 3/36\cdot(5 - 91/36)^2 + 1/36\cdot(6 - 91/36)^2 \approx 1.971$

$\sigma(X) = \text{sqrt}(V(X)) \approx 1.404$

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b) Someone inadvertently steps on one of the two dice, thereby flattening it so that from now on it can only show the numbers one and six (with equal probability). Determine the expected value (no standard deviation) of X after this accident.

X now has the following equally probable 12 values:

1	1	1	1	1	1
1	2	3	4	5	6

$E(X) = 7/12 \cdot 1 + 1/12 \cdot 2 + 1/12 \cdot 3 + 1/12 \cdot 4 + 1/12 \cdot 5 + 1/12 \cdot 6 = 27/12 = 2.25$

c) To make things worse, suddenly a mysterious mechanism comes into effect: The sum of the numbers of the two dice (the flat and the normal one) is never seven any more, while all other possible outcomes, i.e., pairs of numbers, occur with equal probability. What is the new expected value of X?

X now has the following equally probable 10 values:

1	1	1	1	1	
	2	3	4	5	6

$E(X) = 5/10 \cdot 1 + 1/10 \cdot 2 + 1/10 \cdot 3 + 1/10 \cdot 4 + 1/10 \cdot 5 + 1/10 \cdot 6 = 25/10 = 2.5$

d) In the final, mysterious state of the dice, are the numbers that appear on the flat die (X_1) and on the normal die (X_2) independent? Prove your answer.

No, they are not independent. For example, for dice X_1 (the flat one) and X_2 we have:

$p(X_1 = 1) = 5/10 = 0.5$
$p(X_2 = 6) = 1/10 = 0.1$
$p(X_1 = 1 \wedge X_2 = 6) = 0$

Therefore, $p(X_1 = 1) \cdot p(X_2 = 6) \neq p(X_1 = 1 \wedge X_2 = 6)$.

Question 2: The Boston Powerflower

Botanists at UMass Boston recently discovered a new local flower species that they named the Boston powerflower. It has beautiful, blue blossoms, and each flower lives for only one summer. During that time, each flower produces 13 seeds. Three of these seeds will turn into flowers in the following year, and the remaining ten seeds will turn into flowers the year after. As the name powerflower suggests, these seeds always turn into flowers; there is no failure ever.

During the year of this discovery (let us call it year zero), the scientists found three powerflowers on the UMass campus, and in the following year (year one), there were already eight of them.

a) Devise a recurrence relation for the number of flowers f_n in year n, and specify the initial conditions

$f_0 = 3$
$f_1 = 8$
$f_n = 3f_{n-1} + 10f_{n-2}$

b) Use the above recurrence relation to predict the number of flowers on the UMass campus in years 2, 3, 4, and 5.

$f_2 = 3 \cdot 8 + 10 \cdot 3 = 54$

$f_3 = 3 \cdot 54 + 10 \cdot 8 = 242$

$f_4 = 3 \cdot 242 + 10 \cdot 54 = 1266$

$f_5 = 3 \cdot 1266 + 10 \cdot 242 = 6218$

c) Find an explicit formula for computing the number of flowers in any given year without requiring iteration, i.e., repeated application of an equation. You should (but do not have to) check the correctness of your formula using some of the results you obtained in (b).

Characteristic equation:

$r^2 - 3r - 10 = 0$

Roots:

$r_1 = 5$

$r_2 = -2$

Then the solution is of the following form:

$f_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot (-2)^n$

In order to find the values for α_1 and α_2, Let us consider the cases $n = 0$ and $n = 1$, for which we have the initial conditions:

(I) $f_0 = 3 = \alpha_1 \cdot 5^0 + \alpha_2 \cdot (-2)^0 = f_n = \alpha_1 + \alpha_2$

(II) $f_1 = 8 = \alpha_1 \cdot 5 + \alpha_2 \cdot (-2)$

Multiplying Eq. (I) by 2 and adding Eq. (II) to it gives us:

$14 = 7\alpha_1$

Then we easily see that $\alpha_1 = 2$ and $\alpha_2 = 1$, so the solution becomes:

$f_n = 2 \cdot 5^n + (-2)^n$

Let us verify:

$f_2 = 2 \cdot 5^2 + (-2)^2 = 50 + 4 = 54$

$f_3 = 2 \cdot 5^3 + (-2)^3 = 250 - 8 = 242$

$f_4 = 2 \cdot 5^4 + (-2)^4 = 1250 + 16 = 1266$

$f_5 = 2 \cdot 5^5 + (-2)^5 = 6250 - 32 = 6218$

Looks good!

Question 3: Bernoulli and the Red Sox

Let us assume that the probability of the Boston Red Sox to beat the New York Yankees at a baseball game is 60% and that no ties are possible, i.e., the New York Yankees win 40% of the time. Furthermore, the result of each game is independent of the results of any previous games.

a) To collect some money for charity, the Red Sox and Yankees agree to play ten games against each other. What is the probability that the Red Sox win at least eight of these games?

$p(\text{winning 10 games}) = 0.6^{10} \approx 0.006$

$p(\text{winning 9 games}) = C(10, 9) \cdot 0.6^9 \cdot 0.4 \approx 0.0403$

$p(\text{winning 8 games}) = C(10, 8) \cdot 0.6^8 \cdot 0.4^2 \approx 0.1209$

$p(\text{winning at least 8 games}) \approx 0.006 + 0.0403 + 0.1209 \approx 0.1672$

b) In the above charity games, what is the probability that the Yankees win exactly five games?

P(winning 5 games) = C(10, 5)·0.6^5·0.4^5 ≈ 0.2007

c) Later, the Red Sox and Yankees meet again in the playoffs, i.e., they will play games until one team has won four of them. What is the probability that the Red Sox "sweep" the Yankees, i.e., win four games without losing any?

$p = 0.6^4 = 0.1296$

d) What is the probability that the Yankees "sweep" the Red Sox, i.e., win all four games against them without losing any?

$p = 0.4^4 = 0.0256$

e) **Bonus:** What is the probability that the Red Sox beat the Yankees in the playoffs, i.e., are the first to win a total of four games before the Yankees do?

The following things could happen that would lead to the Red Sox winning the playoffs:

- They could win 4 games straight:
  $p_1 = 0.6^4 = 0.1296$

- They could win 3 out of the first 4 games and win the 5th game:
  $p_2 = C(4, 3) \cdot 0.6^3 \cdot 0.4 \cdot 0.6 = 0.2074$

- They could win 3 out of the first 5 games and win the 6th game:
  $p_3 = C(5, 3) \cdot 0.6^3 \cdot 0.4^2 \cdot 0.6 = 0.2074$

- They could win 3 out of the first 6 games and win the 7th game:
  $p_4 = C(6, 3) \cdot 0.6^3 \cdot 0.4^3 \cdot 0.6 = 0.1659$

These cases all exclude each other, so we can compute the overall probability by summing the above 4 values:

$p = p_1 + p_2 + p_3 + p_4 = 0.7103$

f) **Bonus:** Let us assume (the unlikely case) that the Yankees won the first game against the Red Sox in the playoffs. What is the probability that the Red Sox will still beat the Yankees, i.e., win four games before the Yankees win another three?

The following things could happen that would lead to the Red Sox still winning the playoffs:

- They could win 4 games straight:
  $p_1 = 0.6^4 = 0.1296$

- They could win 3 out of the next 4 games and win the 6th game:
  $p_2 = C(4, 3) \cdot 0.6^3 \cdot 0.4 \cdot 0.6 = 0.2074$

- They could win 3 out of the next 5 games and win the 7th game:
  $p_3 = C(5, 3) \cdot 0.6^3 \cdot 0.4^2 \cdot 0.6 = 0.2074$

These cases all exclude each other, so we can compute the overall probability by summing the above 3 values:

$p = p_1 + p_2 + p_3 = 0.5417$

Question 4: Urns and Probabilities

a) An urn contains three red balls and three blue balls. You draw two balls from the urn, one after the other, without putting them back. Event A is that the first ball is red, and event B is that the second ball is red. Are events A and B independent? Prove your answer.

$p(A) = ½$
$p(B) = ½$
$p(A \cap B) = ½ \cdot 2/5 = 1/5$

So we have

$p(A) \cdot p(B) \neq p(A \cap B)$

Therefore, events A and B are not independent.

b) This time you are using the same urn and balls, but you draw three balls, again one after the other. Event C is that the first two balls you draw have different colors, and Event D is that the third ball is blue. Are events C and D independent? Again, prove your answer.

$p(C) = 1/2 \cdot 3/5 + 1/2 \cdot 3/5 = 3/5$
$p(D) = ½$
$p(C \cap D) = ½ \cdot 3/5 \cdot ½ + ½ \cdot 3/5 \cdot ½ = 3/10$

So we have

$p(C) \cdot p(D) = p(C \cap D)$

Therefore, events C and D are independent.