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# Fibonacci Multiplication

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**Abstract.** A curious binary operation on the nonnegative integers is shown to be associative.

A well-known theorem of Zeckendorf [1][3][4] states that every nonnegative integer has a unique representation as a sum of Fibonacci numbers, if we stipulate that no two consecutive Fibonacci numbers occur in the sum. In other words we can uniquely write

$$(1)\qquad n = F_{k_r} + \cdots + F_{k_2} + F_{k_1}\,, \qquad k_r \gg \cdots k_2 \gg k_1 \gg 0, \qquad r \geq 0,$$

where the relation "$k \gg j$" means that $k \geq j+2$. The Fibonacci numbers are defined as usual by the recurrence

$$(2)\qquad F_0 = 0, \qquad F_1 = 1, \qquad F_k = F_{k-1} + F_{k-2} \quad \text{for } k \geq 2.$$

Given the Zeckendorf representations

$$(3)\qquad m = F_{j_q} + \cdots + F_{j_1}\,, \qquad n = F_{k_r} + \cdots + F_{k_1}\,,$$

let us define "circle multiplication" to be the following binary operation:

$$(4)\qquad m \circ n = \sum_{b=1}^{q} \sum_{c=1}^{r} F_{j_b + k_c}\,.$$

In particular, $F_j \circ F_k = F_{j+k}$, if $j \geq 2$ and $k \geq 2$.

The purpose of this note is to prove that circle multiplication satisfies the associative law:

$$(5)\qquad (l \circ m) \circ n = l \circ (m \circ n)\,.$$

The proof is based on a variant of ordinary radix notation that uses Fibonacci numbers instead of powers. Let us write

$$(6)\qquad (d_s \ldots d_1 d_0)_F = d_s F_s + \cdots + d_1 F_1 + d_0 F_0\,.$$

Then $(d_s \ldots d_1 d_0)_F$ is a Zeckendorf representation if and only if the following three conditions hold:

Z1 Each digit $d_i$ is 0 or 1.
Z2 Each pair of adjacent digits satisfies $d_i d_{i+1} = 0$.
Z3 $d_1 = d_0 = 0$.

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For example, here are the Zeckendorf representations for the numbers 1 to 16:

| | |
|---|---|
| $1 = (100)_F$ | $9 = (1000100)_F$ |
| $2 = (1000)_F$ | $10 = (1001000)_F$ |
| $3 = (10000)_F$ | $11 = (1010000)_F$ |
| $4 = (10100)_F$ | $12 = (1010100)_F$ |
| $5 = (100000)_F$ | $13 = (10000000)_F$ |
| $6 = (100100)_F$ | $14 = (10000100)_F$ |
| $7 = (101000)_F$ | $15 = (10001000)_F$ |
| $8 = (1000000)_F$ | $16 = (10010000)_F$ |

Addition of 1 is easy in radix-$F$: We simply set $d_1 d_0 \leftarrow 11$ (which adds 1 to the value) and then use the "carry" rule

$$011 \to 100 \tag{7}$$

as often as possible until there are no two 1's remaining in a row. Finally, we set $d_0 \leftarrow 0$. This procedure makes $d_1 = 0$ after the first carry, so conditions (Z1, Z2, Z3) continue to hold.

In fact, if we begin with any digits $(d_s \ldots d_1 d_0)_F$ that satisfy Z1, we can systematically apply (7) until both Z1 and Z2 are valid. This is obvious because the *binary value* $(d_s \ldots d_1 d_0)_2 = 2^s d_s + \cdots + 2d_1 + d_0$ increases whenever a carry is performed; therefore the process cannot get into a cycle. A given integer has only finitely many representations as a sum of positive Fibonacci numbers, so the process must terminate.

We can also try to add two numbers in radix-$F$ notation, using a variant of ordinary binary addition. First we simply add the digits without carrying; this gives us digits that are 0, 1, or 2. Then we can use the two carry rules

$$0\,(d+1)\,(e+1) \to 1\,d\,e \tag{8}$$

$$0\,(d+2)\,0\,e \to 1\,d\,0\,e \tag{9}$$

to restore the conditions Z1 and Z2.

In fact, we can start with an arbitrary sequence of nonnegative digits $(d_s \ldots d_1 d_0)_F$ and systematically propagate carries by using (8) and (9), always working as far to the left as possible. Each carry increases the binary value, so the process must terminate with a final configuration $(d'_t \ldots d'_1 d'_0)_F$. Since rule (8) is no longer applicable, we must have $d'_i d'_{i+1} = 0$ for all $i \geq 0$. Since rule (9) is no longer applicable, we must also have $d'_i \leq 1$ for all $i \geq 2$.

LEMMA 1. *If $d_i \leq 2$ for all $i \geq 2$ and $d_1 = d_0 = 0$, the carrying process just described transforms $(d_s \ldots d_1 d_0)_F$ into a sequence $(d'_t \ldots d'_1 d'_0)_F$ that satisfies Z1 and Z2.*

PROOF: The result is vacuously true when $s \leq 1$. If $s > 1$, the carrying process applied to $(d_s \ldots d_3\,0\,0)_F$ inductively produces $(d'_t \ldots d'_3 d'_2 d'_1)_F$ with all $d'_i \leq 1$, hence hence

$(d_s \ldots d_3 d_2 d_1 d_0)_F$ is transformed into $(d'_t \ldots d'_3(d_2 + d'_2)d'_1 d_0)_F$. If $d_2 + d'_2 \le 1$ or if $d'_3 = 1$, further carries with (8) will lead to termination without changing $d'_1$. Otherwise we have $d'_3 = 0$ and $2 \le d_2 + d'_2 \le 3$. If $d'_1 = 1$, rule (8) clears $d'_1$; otherwise rule (9) sets $d_0 \leftarrow 1$ and only 0's and 1's remain.

The addition procedure just described is not complete, however, since condition Z3 might not be satisfied. If we can add two numbers without "carrying down" into positions $d_1$ and $d_0$, we say that the addition is *clean*. An unclean addition can be finished up by setting $d'_0 \leftarrow d'_1$ and then carrying if necessary.

Let $\overline{n}$ be the smallest subscript, $k_1$, in the Zeckendorf representation of $n$, when $n > 0$. Thus we have $n = (\ldots 1\, 0 \ldots 0)_F$, with $\overline{n}$ zeroes after the rightmost 1.

LEMMA 2. *If $\overline{m} \ge 4$ and $\overline{n} \ge 4$, the radix-$F$ addition $m + n$ is clean. Moreover, $\overline{m+n} \ge$ $\min(\overline{m}, \overline{n}) - 2$.*

PROOF: Lemma 1 shows that radix-$F$ addition never reduces the number of trailing zeroes by more than 2.

Circle multiplication $m \circ n$ has a natural radix-$F$ interpretation, because it is completely analogous to ordinary binary multiplication. Thus, for example,

$$\begin{aligned} 6 \circ 12 = (100100)_F \circ (1010100)_F &\\ = \qquad (10010000)_F &\\ + (1001000000)_F &\\ + (100100000000)_F & \qquad (10) \end{aligned}$$

because we have $j_2 = 5$, $j_1 = 2$, $k_3 = 6$, $k_2 = 4$, and $k_1 = 2$ in the notation of (3); the three lines of (10) represent $\sum_{b=1}^{2} F_{j_b + k_c}$ for $c = 1, 2, 3$. These are "partial products" $m \circ F_{k_c}$.

Radix-$F$ representation makes it easy to see that circle multiplication is monotonic:

$$l < m \quad \Longrightarrow \quad l \circ n < m \circ n, \qquad \text{for } n > 0. \qquad (11)$$

For if we increase the left factor by 1, every partial product increases.

LEMMA 3. *Radix-$F$ addition of the partial products of $m \circ n$ is clean.*

PROOF: The partial product $m \circ F_k$ has $\overline{m \circ F_k} = \overline{m} + k \ge k + 2$. Since $k_r \gg k_{r-1} \gg$ $k_{r-1} \gg \cdots \gg k_1$, we have, successively,

$$\begin{aligned} \overline{m \circ F_{k_r} + m \circ F_{k_{r-1}}} &\ge k_{r-1}, \\ \overline{m \circ F_{k_r} + m \circ F_{k_{r-1}} + m \circ F_{k_{r-2}}} &\ge k_{r-2}, \\ \ldots \quad \overline{m \circ F_{k_r} + \cdots + m \circ F_{k_1}} &\ge k_1, \end{aligned}$$

by Lemma 2; all of these additions are spanking clean.

THEOREM. *Let the Zeckendorf representations of $l$, $m$, and $n$ be*

$$l = F_{i_p} + \cdots + F_{i_2} + F_{i_1}\,,$$
$$m = F_{j_q} + \cdots + F_{j_2} + F_{j_1}\,,$$
$$n = F_{k_r} + \cdots + F_{k_2} + F_{k_1}\,.$$

*Then the three-fold circle product is the three-fold sum*

$$(l \circ m) \circ n = \sum_{a=1}^{p} \sum_{b=1}^{q} \sum_{c=1}^{r} F_{i_a+j_b+k_c}\,. \tag{12}$$

PROOF: By Lemma 3, each partial product $(l \circ m) \circ F_k$ can be obtained by a clean addition of the partial products $(l \circ F_j) \circ F_k$ of $l \circ m$, shifted left $k$. Hence $(l \circ m) \circ F_k = \sum_{a=1}^{p} \sum_{b=1}^{q} F_{i_a+j_b+k}$, and the result follows by summing over $k = k_1, \ldots, k_r$.

Since the sum in (12) is symmetric in $l$, $m$, and $n$, the circle product must be associative as claimed in (5).

We can extend the proof of Lemma 3 without difficulty to show that the three-fold addition in (12) is clean. Hence we obtain a similar $t$-fold sum for the $t$-fold circle product of any $t$ numbers.

The Fibonacci number $F_k$ is asymptotically $\phi^k/\sqrt{5}$, where $\phi$ is the "golden ratio" $(1 + \sqrt{5})/2$; hence we have

$$F_j \circ F_k \sim \sqrt{5}\, F_j F_k\,, \qquad \text{as } j, k \to \infty. \tag{13}$$

It follows that the circle product $m \circ n$ is approximately $\sqrt{5}\, mn \approx 2.23mn$ when $m$ and $n$ are large. But $1 \circ n$ is closer to $\phi^2 n \approx 2.62n$; and $2 \circ n$ is approximately $\phi^3 n \approx 2.12(2n)$.

This paper was inspired by recent work of Porta and Stolarsky [2], who made the remarkable discovery that the more complicated operation

$$m * n = mn + \lfloor \phi m \rfloor \lfloor \phi n \rfloor$$

is associative. Their "star product" satisfies $m * n \approx 3.62mn$.

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