CS220 – Applied Discrete Mathematics

Relations

March 2, 2021

Relations

- If we want to describe a relationship between elements of two sets A and B, we can use ordered pairs with an element taken from A and an element taken from B.
- Since this is a relation between two sets, it is called a *binary relation*.
- *Definition:* Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.
- In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and a$\underline{\text{R}}$b to denote that $(a, b) \notin R$.

Relations

- When (a, b) belongs to R, a is said to be related to b by R.

  Example: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).
- $P = \{Carl, Suzanne, Peter, Carla\}$
- $C = \{Mercedes, BMW, tricycle\}$
- $D = \{(Carl, Mercedes), (Suzanne, Mercedes),$
  $(Suzanne, BMW), (Peter, tricycle)\}$
- This means that Carl drives a Mercedes, Suzanne drives a Mercedes and a BMW, Peter drives a tricycle, and Carla does not drive any of these vehicles.

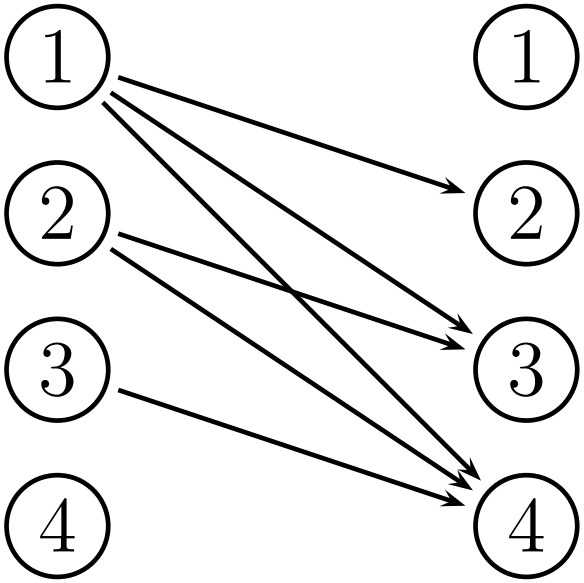
Functions as Relations

- You might remember that a function f from a set A to a set B assigns a unique element of B to each element of A.
- The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$.
- Since the graph of f is a subset of $A \times B$, it is a relation from A to B.
- Moreover, for each element $a \in A$, there is exactly one ordered pair in the graph that has a as its first element.
- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.
- This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

Relations on Sets

- **Definition:** A relation on the set A is a relation from A to A.
- In other words, a relation on the set A is a subset of $A \times A$.
- **Example:** Let $A = \{1, 2, 3, 4\}$.
- Which ordered pairs are in the relation $R = \{(a, b) | a < b\}$?

Relations on Sets



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Relations on Sets

- **Question:** How many different relations can we define on a set A with n elements?
- **Answer:** A relation on a set A is a subset of $A \times A$.
- **Question:** How many elements are in $A \times A$?
- There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?
- The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of $A \times A$.
- **Answer:** We can define 2^{n^2} different relations on A.

Properties of Relations

- We will now look at some useful ways to classify relations.
- **Definition:** A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.
- Are the following relations on $\{1, 2, 3, 4\}$ reflexive?
  1. $R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$
  2. $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$
  3. $R = \{(1, 1), (2, 2), (3, 3)\}$

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  2. $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$
  3. $R = \{(1, 1), (2, 2), (3, 3)\}$
- No, yes, no.
- **Definition:** A relation on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

Definitions:

- A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
- A relation R on a set A is called *antisymmetric* if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.
- A relation R on a set A is called *asymmetric* if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.
- A relation is asymmetric if and only if it is both antisymmetric and irreflexive

Properties of Relations

- **Definition:** A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.
- Are the following relations on $\{1, 2, 3, 4\}$ transitive?
  1. $R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$
  2. $R = \{(1, 3), (3, 2), (2, 1)\}$
  3. $R = \{(2, 4), (4, 3), (2, 3), (4, 1)\}$
- Yes, no, no.

Counting Relations

- **Example:** How many different reflexive relations can be defined on a set A containing n elements?

Counting Relations

- **Example:** How many different reflexive relations can be defined on a set A containing n elements?
- **Solution:** As a reminder, Relations on R are subsets of $A \times A$, which contains n^2 elements.
- Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.
- A reflexive relation, however, must contain the n elements (a, a) for every $a \in A$.
- Consequently, we can only choose among the remaining $n^2 - n = n(n-1)$ elements to generate reflexive relations whether they are in the relation or not, so there are $2^{n(n-1)}$ of them.

Combining Relations

- Relations are sets, and therefore, we can apply the usual set operations to them.
- If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.
- In each case, the result will be another relation from A to B. and there is another important way to combine relations.
- **Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C.
- The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.
- In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then $S \circ R$ contains a pair (a, c).

Combining Relations

- **Example:** Let S and D be relations on $A = \{1, 2, 3, 4\}$.
- $S = \{(a, b)|a < b\}$ – "a is smaller than b"
- $D = \{(a, b)|b = 5 - a\}$ – "b equals (5 - a)"
- $S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
- $D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
- $S \circ D = \{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$
- Remember to work from inside out: D maps an element a to the element $(5 - a)$, and afterwards S maps $(5 - a)$ to all elements larger than $(5 - a)$, resulting in $S \circ D = \{(a, b)|b > 5 - a\}$ or $S \circ D = \{(a, b)|a + b > 5\}$.

Combining Relations

- We already know that functions are just special cases of relations
- Namely, those that map each element in the domain onto exactly one element in the codomain.
- If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).
- **Definition:** Let R be a relation on the set A.
- The powers $R^n, n = 1, 2, 3, \ldots$ are defined inductively by

  $R^1 = R$

  $R^{n+1} = R^n \circ R$
- In other words: $R^n = R \circ R \circ \cdots \circ R$ (n times the letter R)

Combining Relations

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

- Remember the definition of transitivity.
- **Definition:** A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.
- The composite of R with itself contains exactly these pairs (a, c).
- Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.
- Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.

Representing Relations

- We already know different ways of representing relations.
- We will now take a closer look at two ways of representation: Zero-one matrices and directed graphs.
- If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

  $m_{ij} = 1$, if $(a_i, b_j) \in R$, and

  $m_{ij} = 0$, if $(a_i, b_j) \notin R$.
- Note that for creating this matrix we first need to list the elements in A and B in a particular, but arbitrary order.

Representing Relations

- **Example:** How can we represent the relation R from the set $A = \{1, 2, 3\}$ to the set $B = \{1, 2\}$ with $R = \{(2, 1), (3, 1), (3, 2)\}$ as a zero-one matrix?
- **Solution:** The matrix M_R is given by:

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations

- What do we know about the matrices representing a relation on a set (a relation from A to A)?

Representing Relations

- What do we know about the matrices representing a relation on a set (a relation from A to A)?
- They are square matrices.
- What do we know about matrices representing reflexive relations?
- All the elements on the diagonal of such matrices M_{ref} must be 1's.

$$M_{Ref} = \begin{bmatrix} 1 & \cdots & \cdots & \cdots \\ \vdots & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & 1 \end{bmatrix}$$

Representing Relations

- What do we know about the matrices representing symmetric relations?

Representing Relations

- What do we know about the matrices representing symmetric relations?
- These matrices are symmetric, that is, $M_R = (M_R)^T$.
- Which one of these matrices represent a symmetric relation?

$$M_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Representing Relations

- The Boolean operations join and meet (you remember?) can be used to determine the matrices representing the union and the intersection of two relations, respectively.
- To obtain the *join* of two zero-one matrices, we apply the Boolean or function to all corresponding elements in the matrices.
- To obtain the *meet* of two zero-one matrices, we apply the Boolean and function to all corresponding elements in the matrices.

Example

Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- What are the matrices representing $R \cup S$ and $R \cap S$?
- **Solution:** These matrices are given by:

$$M_{(R \cup S)} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{(R \cap S)} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations with Matrices

- Do you remember the *Boolean product* of two zero-one matrices?
- Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix.
- Then the Boolean product of A and B, denoted by $A \circ B$, is the $m \times n$ matrix with $(i, j)^{th}$ entry $[c_{ij}]$, where
- $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$
- $c_{ij} = 1$ if and only if at least one of the terms $(a_{ip} \wedge b_{pj}) = 1$ for some p; otherwise $c_{ij} = 0$.

Representing Relations with Matrices

- Let us now assume that the zero-one matrices $M_A = [a_{ij}]$, $M_B = [b_{ij}]$ and $M_C = [c_{ij}]$ represent relations A, B, and C, respectively.
- In terms of the relations, this means that C contains a pair (x_i, z_j) if and only if there is an element y_p such that (x_i, y_p) is in relation A and (y_p, z_j) is in relation B.
- Therefore, $C = B \circ A$ (composite of A and B).
- This gives us the following rule: $M_{B \circ A} = M_A \circ M_B$
- In other words, the matrix representing the composite of relations A and B is the Boolean product of the matrices representing A and B.
- Analogously, we can find matrices representing the powers of relations: $M_{R^n} = M_R^{[n]}$ (n^{th} Boolean power).

Example

Find the matrix representing R^2, where the matrix representing R is given by:

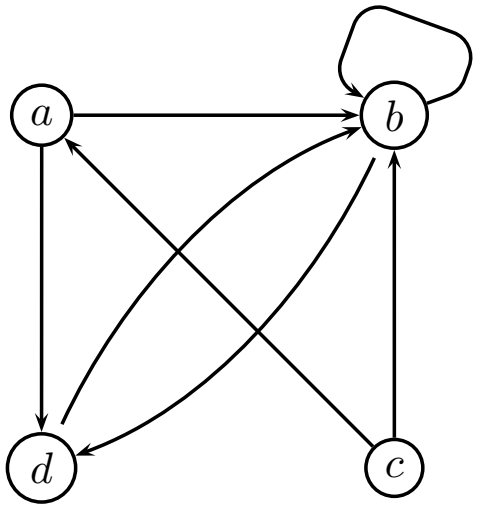
$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The matrix for R^2 is given by:

$$M_{R^2} = {M_R}^2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing Relations with Digraphs

- **Definition:** A *directed graph*, or *digraph*, consists of a set V of *vertices* (or nodes) together with a set E of ordered pairs of elements of V called *edges* (or arcs).
- The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.
- We can use arrows to display digraphs.
- **Example:** Display the digraph with $V = \{a, b, c, d\}$, $E = \{(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)\}$.
- An edge of the form (b, b) is called a *loop*.



Representing Relations with Digraphs

- Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs $(a, b) \in R$ as its edges.
- Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.
- This one-to-one correspondence between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

Closures of Relations

- **Definition:** Let R be a relation on a set A.
- R may or may not have some property P, such as reflexivity, symmetry, or transitivity.
- If there is a relation S that contains R and has property P, and S is a subset of every relation that contains R and has property P, then S is called the closure of R with respect to P.
- Note that the closure of a relation with respect to a property may not exist.

Closures of Relations

- **Example I**: Find the reflexive closure of relation $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on the set $A = \{1,2,3\}$.
- **Solution:** We know that any reflexive relation on A must contain the elements $(1,1)$, $(2,2)$, and $(3,3)$.
- By adding $(2,2)$ and $(3,3)$ to R, we obtain the reflexive relation S, which is given by
- $S = \{(1,1),(1,2),(2,1),(2,2),(3,2),(3,3)\}$.
- S is reflexive, contains R, and is contained within every reflexive relation that contains R.
- Therefore, S is the reflexive closure of R.

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- **Example III:** Find the transitive closure of the relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3, 4\}$.

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- **Solution:** R would be transitive, if for all pairs (a, b) and (b, c) in R there were also a pair (a, c) in R.
- If we add the missing pairs $(1, 2), (2, 3), (2, 4), (3, 1)$, will R be transitive?

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- **Solution:** R would be transitive, if for all pairs (a, b) and (b, c) in R there were also a pair (a, c) in R.
- If we add the missing pairs $(1, 2), (2, 3), (2, 4), (3, 1)$, will R be transitive?
- No, because the extended relation R contains $(3, 1)$ and $(1, 4)$, but does not contain $(3, 4)$.
- By adding new elements to R, we also add new requirements for its transitivity.
- We need to look at paths in digraphs to solve this problem.

Closures of Relations

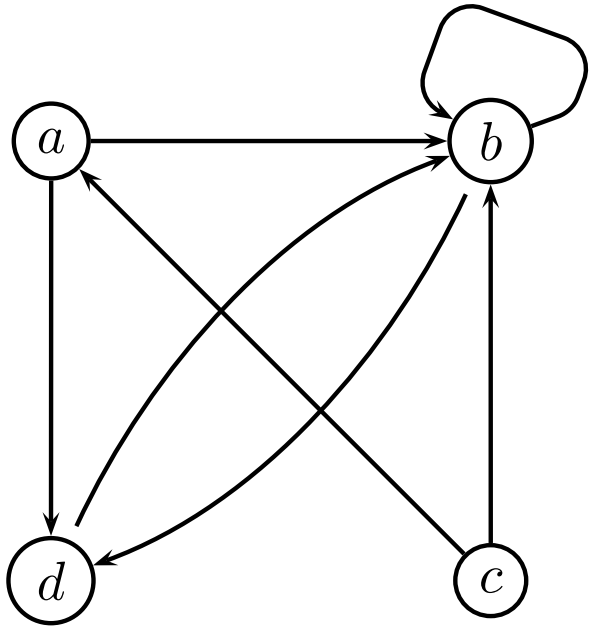
- Imagine that we have a relation R that represents all **train connections** in the US.
- For example, if (Boston, Philadelphia) is in R, then there is a direct train connection from Boston to Philadelphia.
- If R contains (Boston, Philadelphia) and (Philadelphia, Washington), there is an indirect connection from Boston to Washington.
- Because there are indirect connections, it is not possible by just looking at R to determine which cities are connected by trains.
- The transitive closure of R contains exactly those pairs of cities that are connected, either directly or indirectly.

Closures of Relations

- **Definition:** A *path* from a to b in the directed graph G is a sequence of one or more edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where $x_0 = a$ and $x_n = b$.
- In other words, a path is a sequence of edges where the terminal vertex of an edge is the same as the initial vertex of the next edge in the path.
- This path is denoted by $x_0, x_1, x_2, \ldots, x_n$ and has length n.
- A path that begins and ends at the same vertex is called a *circuit* or *cycle*.

Closures of Relations

See example from before:



- Is **c,a,b,d,b** a path in this graph?
- Is **d,b,b,b,d,b,d** a cycle in this graph?
- Is there any cycle including **c** in this graph?

Closures of Relations

- Due to the one-to-one correspondence between graphs and relations, we can transfer the definition of path from graphs to relations:
- **Definition:** There is a path from a to b in a relation R, if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \ldots (x_{n-1}, b) \in R$.

Theorem

Let R be a relation on a set A. There is a path from a to b if and only if $(a, b) \in R^n$ for some positive integer n.

Closures of Relations

- According to the train example, the transitive closure of a relation consists of the pairs of vertices in the associated directed graph that are connected by a path.
- **Definition:** Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path between a and b in R.
- We know that R^n consists of the pairs (a, b) such that a and b are connected by a path of length n.
- Therefore, R^* is the union of R^n across all positive integers n:

$$R^* = \bigcup_{n=1}^{\infty} R^n = R^1 \cup R^2 \cup R^3 \ldots$$

Closures of Relations

Theorem

The transitive closure of a relation R equals the connectivity relation R^*.

- But how can we compute R^* ?

Lemma

Let A be a set with n elements, and let R be a relation on A. If there is a path in R from a to b, then there is such a path with length not exceeding n.

- Moreover, if $a \neq b$ and there is a path in R from a to b, then there is such a path with length not exceeding $(n-1)$.

Closures of Relations

- This lemma is based on the observation that if a path from a to b visits any vertex more than once, it must include at least one cycle.
- These cycles can be eliminated from the path, and the reduced path will still connect a and b.

Theorem

For a relation R on a set A with n elements, the transitive closure R^* is given by: $R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

- For matrices representing relations we have: $M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}} \vee \cdots \vee M_{R^{[n]}}$

- Let us finally solve **Example III** by finding the transitive closure of the relation $R = \{(1,3),(1,4),(2,1),(3,2)\}$ on the set $A = \{1,2,3,4\}$.
- R can be represented by the following matrix M_R:

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closures of Relations

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[3]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_R^{[4]} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R* = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Or the relation:

$$\{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),$$
$$(3,1),(3,2),(3,3),(3,4)\}$$

Partial Orderings

- Sometimes, relations define an order on the elements in a set.
- **Definition:** A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive.
- A set S together with a partial ordering R is called a *partially ordered* set, or *poset*, and is denoted by (S, R).
- **Example:** Consider the "greater than or equal" relation $\geq$ (defined by $\{(a, b)|a \geq b\}$).
- Is $\geq$ a partial ordering on the set of integers?
- $\geq$ is reflexive, because $a \geq a$ for every integer a.
- $\geq$ is antisymmetric, because if $a \neq b$, then $a \geq b \wedge b \geq a$ is false.
- $\geq$ is transitive, because if $a \geq b$ and $b \geq c$, then $a \geq c$.
- Consequently, $(\mathbb{Z}, \geq)$ is a partially ordered set.

Partial Orderings

- Example: Another example: Is the "inclusion relation" $\subseteq$ a partial ordering on the power set of a set S?
- $\subseteq$ is reflexive, because $A \subseteq A$ for every set A.
- $\subseteq$ is antisymmetric, because if $A \neq B$, then $A \subseteq B \wedge B \subseteq A$ is false.
- $\subseteq$ is transitive, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Consequently, $(P(S), \subseteq)$ is a partially ordered set.
- In a poset the notation $a \leq b$ denotes that $(a, b) \in R$.
- Note that the symbol $\leq$ is used to denote the relation in any poset, not just the "less than or equal" relation.
- The notation $a < b$ denotes that $a \leq b$, but $a \neq b$.
- If $a \leq b$ we say "a is less than b" or "b is greater than a".

Partial Orderings

- For two elements a and b of a poset $(S, \leq)$ it is possible that neither $a \leq b$ nor $b \leq a$.
- **Example:** In $(P(\mathbb{Z}), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, since neither is contained within the other.
- **Definition:** The elements a and b of a poset $(S, \leq)$ are called *comparable* if either $a \leq b$ or $b \leq a$.
- When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$, then a and b are called *incomparable*.
- For some applications, we require all elements of a set to be comparable.
- For example, if we want to write a dictionary, we need to define an order on all English words (alphabetic order).
- **Definition:** If $(S, \leq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or linearly ordered set, and $\leq$ is called a *total order* or *linear order*.
- A totally ordered set is also called a *chain*.

Partial Orderings

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- **Example II:** Is $(\mathbb{Z}^+, |)$ a totally ordered set? (a divides b)
- No, because it contains incomparable elements such as 5 and 7.

Equivalence Relation

- *Equivalence relations* are used to relate objects that are similar in some way.
- **Definition:** A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.
- Two elements that are related by an equivalence relation R are called *equivalent*.
- Since R is symmetric, a is equivalent to b whenever b is equivalent to a.
- Since R is reflexive, every element is equivalent to itself.
- Since R is transitive, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.
- Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relation

- **Example:** Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x.
- Is R an equivalence relation?

Equivalence Relation

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- Is R an equivalence relation?

Solution:

- R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a.
- R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.
- R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.
- Therefore R is an equivalence relation.

Equivalence Classes

- **Definition:** Let R be an equivalence relation on a set A.
- The set of all elements that are related to an element a of A is called the *equivalence class* of a.
- The equivalence class of a with respect to R is denoted by $[a]_R$.
- When only one relation is under consideration, we will delete the subscript R and write $[a]$ for this equivalence class.
- If $b \in [a]_R$, b is called a *representative* of this equivalence class.

Equivalence Classes

- **Example:** In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?

Equivalence Classes

- **Example:** In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?
- Solution: [mouse] is the set of all English words containing five letters.
- For example, 'horse' would be a representative of this equivalence class.

Equivalence Classes

Theorem

Let R be an equivalence relation on a set A. The following statements are equivalent:

1. aRb
2. $[a] = [b]$
3. $[a] \cap [b] \neq \varnothing$

- **Reminder:** A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union.
- In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if
  1. $A_i \neq \varnothing$ for $i \in I$
  2. $A_i \cap A_j = \varnothing$, if $i \neq j$
  3. $\bigcup_{i \in I} A_i = S$

Equivalence Classes

Theorem

- *Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.*
- *Conversely, given a partition $\{A_i | i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.*

Equivalence Classes

- **Example:** Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.
- Let R be the equivalence relation $\{(a, b)|$a and b live in the same city$\}$ on the set $P = \{Frank, Suzanne, George, Stephanie, Max, Jennifer\}$.
- Then $R = \{$(Frank, Frank), (Frank, Suzanne), (Frank, George), (Suzanne, Frank), (Suzanne, Suzanne), (Suzanne, George), (George, Frank), (George, Suzanne), (George, George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max, Max), (Jennifer, Jennifer)$\}$.
- Then the equivalence classes of R are the following partition: $\{\{Frank, Suzanne, George\}, \{Stephanie, Max\}, \{Jennifer\}\}$
- The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to exactly one of the equivalence classes.

Equivalence Classes

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Equivalence Classes

- **Example:** Let R be the relation $\{(a, b)|a \equiv b(mod 3)\}$ on the set of integers.
- Is R an equivalence relation?
- Yes, R is reflexive, symmetric, and transitive.
- What are the equivalence classes of R ?

  $\{\{\ldots, -6, -3, 0, 3, 6, \ldots\},$

  $\{\ldots, -5, -2, 1, 4, 7, \ldots\},$

  $\{\ldots, -4, -1, 2, 5, 8, \ldots\}\}$
- Again, these three classes form a partition of the set of integers.

N-ary Relations

- In order to study an interesting application of relations, namely databases, we first need to generalize the concept of binary relations to n-ary relations.
- **Definition:** Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$.
- The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is called its degree.
- **Example:**
- Let $R = \{(a, b, c) | a = 2b \wedge b = 2c \text{ with } a, b, c \in \mathbb{Z}\}$
- The degree of R is 3, so its elements are triples.
- Its domains are all equal to the set of integers.
- Is $(2, 4, 8)$ in R? No.
- Is $(4, 2, 1)$ in R? Yes.

Relations and Databases

- Let us take a look at a type of database representation that is based on relations, namely the *relational data model*.
- A database consists of n-tuples called *records*, which are made up of *fields*.
- These fields are the entries of the n-tuples.
- The relational data model represents a database as an n-ary relation, that is, a set of records.

Relations and Databases

Example: Consider a database of students, whose records are represented as 4-tuples with the fields Student Name, ID Number, Major, and GPA:

$R = \{(Ackermann, 00231455, CS, 3.88),$

$(Adams, 00888323, Physics, 3.45),$

$(Chou, 00102147, CS, 3.79),$

$(Goodfriend, 00453876, Math, 3.45),$

$(Rao, 00678543, Math, 3.90),$

$(Stevens, 00786576, Psych, 2.99)\}$

- Relations that represent databases are also called *tables*, since they are often displayed as tables.

Relations and Databases

- In a database, a primary key should remain a primary key even if new records are added.
- Therefore, we should use a primary key of the *intension* of the database, containing all the n-tuples that can ever be included in our database.
- Combinations of domains can also uniquely identify n-tuples in an n-ary relation.
- When the values of a set of domains determine an n-tuple in a relation, the Cartesian product of these domains is called a *composite key*.
- We can apply a variety of operations on n-ary relations to form new relations.

Relations and Databases

- **Definition:** The *projection* $P_{i_1,i_2,\ldots,i_m}$ maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.
- In other words, a projection $P_{i_1,i_2,\ldots,i_m}$ keeps the m components $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ of an n-tuple and deletes its $(n - m)$ other components.
- **Example:** What is the result when we apply the projection $P_{2,4}$ to the student record $Stevens, 00786576, Psych, 2.99)$?
- Solution: It is the pair $(00786576, 2.99)$.
- In some cases, applying a projection to an entire table may not only result in fewer columns, but also in fewer rows.
- Why is that?
- Some records may only have differed in those fields that were deleted, so they become identical, and there is no need to list identical records more than once.

Relations and Databases

- We can use the *join* operation to combine two tables into one if they share some identical fields.
- **Definition:** Let R be a relation of degree m and S a relation of degree n.
- The *join* $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all (m + n - p)-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.
- In other words, to generate $J_p(R, S)$, we have to find all the elements in R whose p last components match the p first components of an element in S.
- The new relation contains exactly these matches, which are combined to tuples that contain each matching field only once.

Example: What is $J_1(Y, R)$, where Y contains the fields Student Name and Year of Birth,

$$\begin{aligned} Y = \{&(1978, \textit{Ackermann}),\\ &(1994, \textit{Adams}),\\ &(1997, \textit{Chou}),\\ &(1984, \textit{Goodfriend}),\\ &(1982, \textit{Rao})\\ &(1970, \textit{Stevens})\} \end{aligned}$$

- and R contains the student records as defined before?

- **Solution:** The resulting relation is:

  $\{(1978, \textit{Ackermann}, 00231455, \textit{CS}, 3.88),$
  $(1994, \textit{Adams}, 00888323, \textit{Physics}, 3.45),$
  $(1997, \textit{Chou}, 00102147, \textit{CS}, 3.79),$
  $(1984, \textit{Goodfriend}, 00453876, \textit{Math}, 3.45),$
  $(1982, \textit{Rao}, 00678543, \textit{Math}, 3.90),$
  $(1970, \textit{Stevens}, 00786576, \textit{Psych}, 2.99)\}$

- Since Y has two fields and R has four, the relation $J_1(Y, R)$ has $2 + 4 - 1 = 5$ fields.