Propositional Logic and Boolean Algebra

CS 220 — Applied Discrete Mathematics

September 10, 2024

Statements

Definition (Statement)

A **statement** is an unambiguous, declarative sentence that is either objectively true or false.

Examples (Statements)

- It is raining, and I have no umbrella.
- Either that is a cat, or I forgot to turn off the oven.
- Sodium hydroxide is an ingredient in solid soaps.
- The floor is lava.
- There is another planet with intelligent life within 200 light-years of Earth.

Examples (Not statements)

- The smallest two-digit prime number.
- Who knows what evil lurks in the hearts of men?
- Look at that plumage!
- 2 is the best number. *
- "Embiggen" is a perfectly cromulent word. *

Statement Ingredients

Observation

A statement seems to consist of

- **domain knowledge** (math, science, programming, etc)
- **logical structure**

Examples (Domain knowledge)

- "It is raining."
- "7 is odd."
- "Sodium hydroxide is an ingredient in solid soaps."
- "Array A has 5 elements."

Examples (Logical structure)

- "______ and ______."
- "______ or ______."
- "If ______, then ______."
- "It's not true that ______."

Idea: let's split them and tackle them separately.

Exercise: Decompose Statements

Logical structures:

"________ and ________." "________ or ________."
"If ________, then ________." "It's not true that ________."

Examples

Decompose each statement into basic statements and logical structure.

1. "It is raining, and I have an umbrella."
2. "If it is snowing, then the trees have flowers."
3. "If $f(x) > a$, then $f(x+1) > 2a$."
4. "36 is a multiple of either 8 or 9."
5. "2 is less than 3, which is less than 5."
6. "I like apples and oranges, but not pears."

Exercise: Decompose Statements

Logical structures:

"_______ and _______." "_______ or _______."
"If _______, then _______." "It's not true that _______."

Examples

Decompose each statement into *basic statements* and **logical structure**.

1. "*It is raining*, **and** *I have an umbrella*."
2. "**If** *it is snowing*, **then** *the trees have flowers*."
3. "**If** $f(x) > a$, **then** $f(x+1) > 2a$."
4. "36 is a multiple of either 8 or 9."
   = "*36 is a multiple of 8*, **or** *36 is a multiple of 9*."
5. "2 is less than 3, which is less than 5."
   = "*2 is less than 3*, **and** *3 is less than 5*."
6. "I like apples and oranges, but not pears."
   = "*I like apples*, **and** *I like oranges*, **and it is not true that** *I like pears*."

Propositional Logic

Definition (Propositional Logic)

In **propositional logic** there are two kinds of **propositions**:

- **propositional variables** (aka, **atomic propositions**) which stand for individual statements of domain knowledge, and
- **compound propositions** formed by combining smaller propositions with **logical connectives** (aka, **logical operators**)

A proposition is a formal representation of a statement.

Example

Let R represent "it is raining" and let U represent "I have an umbrella".

We write $\neg U$ for "I do *not* have an umbrella".

We write $R \wedge \neg U$ for "It is raining, *and* I do not have an umbrella".

Compound Propositions

Here are the logical connectives most used in mathematics:

Connective	Read as	Preferred notation	Other notations	
Negation	"not"	$\neg P$	$\sim P$	$!P$
Conjunction	"and"	$P \wedge Q$	$P \& Q$	
Disjunction	"or"	$P \vee Q$		
Implication	"implies", "if–then"	$P \Rightarrow Q$	$P \rightarrow Q$	$P \supset Q$
Biconditional	"equivalent to"	$P \Leftrightarrow Q$	$P \leftrightarrow Q$	$P \equiv Q$

Other logical operators are used in other contexts; for example, NAND and NOR and XOR are common in digital circuit design.

Evaluating the Truth of Propositions

Definition (Truth Value)

There are two **truth values**: true (T) and false (F).

The truth value of a compound proposition depends only on

- the logical connective, and
- the truth values of the component propositions

It does *not* depend on the statements themselves.

For example, we evaluate these propositions in exactly the same way:

- "8 is even" $\Rightarrow$ "9 is odd"
- "8 is even" $\Rightarrow$ "Mars is a planet"

That is, the logical connectives simply act like operators on truth values.
Their behavior can be summarized by **truth tables**.

Negation ($\neg$, "not", "it is not true that")

P	$\neg P$
T	F
F	T

Examples

- "I don't like pears" $=$ $\neg$("I like pears") $=$ $\neg$F $=$ T
- "today is not Tuesday" $=$ $\neg$("today is Tuesday") $=$ $\neg$T $=$ F

Conjunction ($\wedge$, “and”)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples

- “$2 < 3$ and $3 < 5$” $=$ $(2 < 3) \wedge (3 < 5)$ $=$ T $\wedge$ T $=$ T
- “it is sunny, and I have an umbrella”
  $=$ (“it is sunny”) $\wedge$ (“I have an umbrella”) $=$ T $\wedge$ F $=$ F

Disjunction ($\vee$, "or")

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Notice that it is an inclusive "or", not always what we mean in a spoken language. (Not like: "You can have soup *or* salad.")

Examples

- "it is raining or the sprinklers are on"
  $=$ ("it is raining") $\vee$ ("the sprinklers are on") $=$ F $\vee$ T $=$ T
- "72 is a multiple of 6 or a multiple of 8"
  $=$ ("72 is a multiple of 6") $\vee$ ("72 is a multiple of 8") $=$ T $\vee$ T $=$ T

Exclusive Or ($\oplus$, XOR)

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

- It is true if *exactly* one of the predicates is true.

Implication ($\Rightarrow$, "if–then", "implies")

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- P is called the **hypothesis**; Q is called the **conclusion**.
- If P is false, then $P \Rightarrow Q$ is true regardless of Q.
- $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$.

Examples

- "if today is Tuesday, then you have class"
- "if my light is green then the other light is red"

Implication ($\Rightarrow$, “if–then”, “implies”)

Definition (Contrapositive)

The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

The contrapositive is equivalent to the original proposition.

Examples

- **Original:** “If today is Tuesday, then you have class.”
  Contrapositive: “If you don’t have class, then today is not Tuesday.”
- **Original:** “If my light is green, then the other light is red.”
  Contrapositive: “If the other light is not red, then my light is not green.”

Biconditional ("if and only if", sometimes written "iff")

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

- True if both P and Q have the same truth value.
- Equivalent to $(P \Rightarrow Q) \land (Q \Rightarrow P)$.

Summary: Logical Connectives

P	$\neg P$
T	F
F	T

P	Q	$P \land Q$	$P \lor Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Evaluating Propositions

Truth tables can be used to evaluate complex propositions.

1. Create a column for each propositional variable, and create $2^{\text{\#vars}}$ rows. Fill in every combination of truth values for the propositional variables.
2. Create a column for each sub-expression, smallest first.
3. Fill in each column by applying the truth table of its main connective to the truth values of its arguments *from that row*.

Example

Evaluate the proposition: $\neg P \vee \neg Q$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Evaluating Multiple Propositions

You can evaluate multiple statements in a truth table.

Example

Evaluate $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$.

Evaluating Multiple Propositions

You can evaluate multiple statements in a truth table.

Example

Evaluate $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Evaluating Multiple Propositions

You can evaluate multiple statements in a truth table.

Example

Evaluate $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Notice that $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ have the same truth table.
That is, they are **logically equivalent**.

Logically Equivalent Statements

If two propositions X and Y are logically equivalent, then $X \Leftrightarrow Y$ is always T, and vice versa.

Example

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$
T	T	F	F	T
F	T	T	T	T
T	F	T	T	T
F	F	T	T	T

Logical Equivalences

Name	Disjunction	Conjunction
Identity	$A \vee \mathsf{F} \;\Leftrightarrow\; A$	$A \wedge \mathsf{T} \;\Leftrightarrow\; A$
Dominance	$A \vee \mathsf{T} \;\Leftrightarrow\; \mathsf{T}$	$A \wedge \mathsf{F} \;\Leftrightarrow\; \mathsf{F}$
Idempotent	$A \vee A \;\Leftrightarrow\; A$	$A \wedge A \;\Leftrightarrow\; A$
Inverse	$A \vee \neg A \;\Leftrightarrow\; \mathsf{T}$	$A \wedge \neg A \;\Leftrightarrow\; \mathsf{F}$
Commutative	$A \vee B \;\Leftrightarrow\; B \vee A$	$A \wedge B \;\Leftrightarrow\; B \wedge A$
Associative	$(A \vee B) \vee C \;\Leftrightarrow\; A \vee (B \vee C)$	$(A \wedge B) \wedge C \;\Leftrightarrow\; A \wedge (B \wedge C)$
Distributive	$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
Absorption	$A \vee (A \wedge B) \;\Leftrightarrow\; A$	$A \wedge (A \vee B) \;\Leftrightarrow\; A$
DeMorgan	$\neg(A \vee B) \;\Leftrightarrow\; \neg A \wedge \neg B$	$\neg(A \wedge B) \;\Leftrightarrow\; \neg A \vee \neg B$

Logical Equivalences

Name	**Equivalence**
Double Negation	$\neg(\neg A) \;\Leftrightarrow\; A$
Conditional	$A \Rightarrow B \;\Leftrightarrow\; \neg A \vee B$
Contrapositive	$A \Rightarrow B \;\Leftrightarrow\; \neg B \Rightarrow \neg A$
Biconditional	$(A \Leftrightarrow B) \;\Leftrightarrow\; ((A \Rightarrow B) \wedge (B \Rightarrow A))$

Tautologies and Contradictions

Definitions (Tautology, Contradiction)

A **tautology** is a proposition that is always true.

A **contradiction** is a proposition that is always false.

A **contingent proposition** is neither always true nor always false.
Its truth value depends on the truth values of its propositional variables.

Examples (Tautologies)

- $R \lor \neg R$
- $\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

Examples (Contradictions)

- $R \land \neg R$
- $(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

The negation of any tautology is a contradiction.
The negation of any contradiction is a tautology.
The negation of any contingent proposition is contingent.

Satisfiability

Definition (Truth Assignment)

A **truth assignment** maps propositional variables to truth values.
Each row of a truth table corresponds to a truth assignment.

Example

There's no standard notation, but we could write $\{A = \mathsf{T}, B = \mathsf{F}\}$.

Definition (Satisfiable)

A proposition is **satisfiable** if there is a truth assignment that makes it true.
A proposition is **valid** if every truth assignment makes it true.
A proposition is **unsatisfiable** if every truth assignment makes it false.
(That is, valid = tautology, unsatisfiable = contradiction.)

Is there an algorithm for determining if a proposition is satisfiable?
Is there an algorithm for determining if a proposition is valid?

From Truth Table to Proposition

Given a truth table, can we find a proposition with that table?

Example

P	Q	R	?
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

From Truth Table to Proposition

Given a truth table, can we find a proposition with that table?

Example

P	Q	R	?
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

We can get a true result by:

picking row 4 OR
picking row 7 OR
picking row 8

Under what circumstances does row 4 apply? (Likewise, 7 and 8.)

From Truth Table to Proposition

Given a truth table, can we find a proposition with that table?

Example

P	Q	R	?	
T	T	T	F	
T	T	F	F	
T	F	T	F	
T	F	F	T	$P \land \neg Q \land \neg R$
F	T	T	F	
F	T	F	F	
F	F	T	T	$\neg P \land \neg Q \land R$
F	F	F	T	$\neg P \land \neg Q \land \neg R$

We can get a true result by:

picking row 4 OR
picking row 7 OR
picking row 8

Under what circumstances does row 4 apply? (Likewise, 7 and 8.)

From Truth Table to Proposition

Given a truth table, can we find a proposition with that table?

Example

P	Q	R	?	
T	T	T	F	
T	T	F	F	
T	F	T	F	
T	F	F	T	$P \wedge \neg Q \wedge \neg R$
F	T	T	F	
F	T	F	F	
F	F	T	T	$\neg P \wedge \neg Q \wedge R$
F	F	F	T	$\neg P \wedge \neg Q \wedge \neg R$

We can get a true result by:

picking row 4 OR
picking row 7 OR
picking row 8

Under what circumstances does row 4 apply? (Likewise, 7 and 8.)

Solution:

$$[P \wedge \neg Q \wedge \neg R] \;\vee\; [\neg P \wedge \neg Q \wedge R] \;\vee\; [\neg P \wedge \neg Q \wedge \neg R]$$

This proposition is in disjunctive normal form.

Rewriting with Logical Equivalences

That technique gives us *some* proposition. Is it the best? The shortest?

We can rewrite the proposition using logical equivalences:

$$
\begin{array}{rlr}
& [P \land \neg Q \land \neg R] \;\lor\; [\neg P \land \neg Q \land R] \;\lor\; [\neg P \land \neg Q \land \neg R] & \\
= & \neg Q \;\land\; [(P \land \neg R) \;\lor\; (\neg P \land R) \;\lor\; (\neg P \land \neg R)] & \text{Distrib.} \\
= & \neg Q \;\land\; [(P \land \neg R) \;\lor\; (\neg P \land \neg R) \;\lor\; (\neg P \land R)] & \text{Commut.} \\
= & \neg Q \;\land\; [(P \land \neg R) \;\lor\; (\neg P \land \neg R) \;\lor\; (\neg P \land \neg R) \;\lor\; (\neg P \land R)] & \text{Idem.} \\
= & \neg Q \;\land\; [((P \lor \neg P) \land \neg R) \;\lor\; (\neg P \land (\neg R \lor R))] & \text{Distrib.} \\
= & \neg Q \;\land\; [(\mathsf{T} \land \neg R) \;\lor\; (\neg P \land \mathsf{T})] & \text{Inv.} \\
= & \neg Q \;\land\; [\neg R \;\lor\; \neg P] & \text{Ident.} \\
= & \neg Q \;\land\; \neg(R \land P) & \text{DeMorgan} \\
= & \neg(Q \lor (R \land P)) & \text{DeMorgan}
\end{array}
$$

This should remind you of algebra. It’s almost the same.

Boolean Algebra

Pretend we had “ordinary” variables $x, \overline{x}, y, \overline{y}, z, \overline{z}$ instead of propositional variables and their negations. This part is familiar:

$$
\begin{aligned}
& [x \cdot \overline{y} \cdot \overline{z}] \; + \; [\overline{x} \cdot \overline{y} \cdot z] \; + \; [\overline{x} \cdot \overline{y} \cdot \overline{z}] && \\
= \; & \overline{y} \; \cdot \; [(x \cdot \overline{z}) \; + \; (\overline{x} \cdot z) \; + \; (\overline{x} \cdot \overline{z})] && \text{Distrib.} \\
= \; & \overline{y} \; \cdot \; [(x \cdot \overline{z}) \; + \; (\overline{x} \cdot \overline{z}) \; + \; (\overline{x} \cdot z)] && \text{Commut.} \\
= \; & \overline{y} \; \cdot \; [(x \cdot \overline{z}) \; + \; (\overline{x} \cdot \overline{z}) \; + \; (\overline{x} \cdot \overline{z}) \; + \; (\overline{x} \cdot z)] && \text{Idem.} \\
= \; & \overline{y} \; \cdot \; [((x + \overline{x}) \cdot \overline{z}) \; + \; (\overline{x} \cdot (\overline{z} + z))] && \text{Distrib.}
\end{aligned}
$$

Now we have to introduce some special rules to deal with those bars:

$$
\begin{aligned}
= \; & \overline{y} \; \cdot \; [(1 \cdot \overline{z}) \; + \; (\overline{x} \cdot 1)] && \text{Inv.} \\
= \; & \overline{y} \; \cdot \; [\overline{z} \; + \; \overline{x}] && \text{Ident.} \\
= \; & \overline{y} \; \cdot \; \overline{(z \cdot x)} && \text{DeMorgan} \\
= \; & \overline{(y + (z \cdot x))} && \text{DeMorgan}
\end{aligned}
$$

Boolean Algebra

Parts of propositional logic obey rules very similar to traditional algebra:

P, Q, R	are like	x, y, z
T and F	are like	1 and 0
$P \wedge Q$	is like	$x \cdot y$
$P \vee Q$	is like	$x + y$
$\overline{P}$	is like	$1 - x$
$\mathsf{T} \wedge \mathsf{F} = \mathsf{F}$	is like	$1 \cdot 0 = 0$

But there are some important exceptions.

- $\mathsf{T} \vee \mathsf{T} = \mathsf{T}$ so $1 + 1 = 1$ (!!)
- "addition" distributes over "multiplication"
- ...

The algebraic formulation of propositional logic is called **Boolean algebra**.
(We'll come back to this.)