Propositional Logic

CS 220 — Applied Discrete Mathematics

February {3, 5}, 2025

UMass Boston

Statements

Definition (Statement)

A **statement** is an unambiguous, declarative sentence that is objectively true or false.

Examples (Statements)

- It is raining, and I have no umbrella.
- If the wait-queue is empty, the machine halts.
- Sodium hydroxide is an ingredient in solid soaps.
- The floor is lava.
- There is another planet with intelligent life within 200 light-years of Earth.

Examples (Not statements)

- The smallest two-digit prime number.
- Who knows what evil lurks in the hearts of men?
- Look at that plumage!
- 2 is the best number. *
- "Embiggen" is a perfectly cromulent word. *

Statement Ingredients

Observation

A statement seems to consist of

- **domain knowledge** (math, science, programming, etc)
- **logical structure**

Examples (Domain knowledge)

- "It is raining."
- "7 is odd."
- "Sodium hydroxide is an ingredient in solid soaps."
- "Array A has 5 elements."

Examples (Logical structure)

- "_______ and _______."
- "_______ or _______."
- "If _______, then _______."
- "It's not true that _______."

Idea: let's split them and tackle them separately.

Exercise: Decompose Statements

Logical structures:

"________ and ________." "________ or ________."
"If ________, then ________." "It's not true that ________."

Examples

Decompose each statement into basic statements and logical structure.

1. "It is raining, and I have an umbrella."
2. "If it is snowing, then the trees have flowers."
3. "If $f(x) > a$, then $f(x+1) > 2a$."
4. "36 is a multiple of either 8 or 9."
5. "2 is less than 3, which is less than 5."
6. "I like apples and oranges, but not pears."

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"If ________, then ________."   "It's not true that ________."

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Decompose each statement into *basic statements* and **logical structure**.

1. "*It is raining*, **and** *I have an umbrella*."
2. "**If** *it is snowing*, **then** *the trees have flowers*."
3. "**If** $f(x) > a$, **then** $f(x+1) > 2a$."
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1. "*It is raining*, **and** *I have an umbrella*."
2. "**If** *it is snowing*, **then** *the trees have flowers*."
3. "**If** $f(x) > a$, **then** $f(x+1) > 2a$."
4. "36 is a multiple of either 8 or 9."
   = "*36 is a multiple of 8*, **or** *36 is a multiple of 9*."
5. "2 is less than 3, which is less than 5."
   = "*2 is less than 3*, **and** *3 is less than 5*."
6. "I like apples and oranges, but not pears."
   = "*I like apples*, **and** *I like oranges*, **and it is not true that** *I like pears*."

Propositional Logic

Definition (Propositional Logic)

In **propositional logic** there are two kinds of **propositions**:

- **propositional variables** (aka, **atomic propositions**) which stand for individual statements of domain knowledge, and
- **compound propositions** formed by combining smaller propositions with **logical connectives** (aka, **logical operators**)

A proposition is a formal representation of a statement.

Example

Let R represent “it is raining” and let U represent “I have an umbrella”.

We write $\neg U$ for “I do *not* have an umbrella”.

We write $R \wedge \neg U$ for “It is raining, *and* I do not have an umbrella”.

Compound Propositions

Here are the logical connectives most used in mathematics:

Connective	Read as	Preferred notation	Other notations
Negation	"not"	$\neg P$	$\sim P$ $\quad !P$
Conjunction	"and"	$P \wedge Q$	$P \& Q$
Disjunction	"or"	$P \vee Q$	
Implication	"implies", "if–then"	$P \Rightarrow Q$	$P \rightarrow Q$ $\quad P \supset Q$
Biconditional	"equivalent to"	$P \Leftrightarrow Q$	$P \leftrightarrow Q$ $\quad P \equiv Q$

Other logical operators are used in other contexts; for example, NAND and NOR and XOR are common in digital circuit design.

Evaluating the Truth of Propositions

Definition (Truth Value)

There are two **truth values**: true (T) and false (F).

The truth value of a compound proposition depends only on

- the logical connective, and
- the truth values of the component propositions

For example, we evaluate these propositions in exactly the same way:

- "8 is even" $\Rightarrow$ "9 is odd" T
- "8 is even" $\Rightarrow$ "Mars is a planet" T

That is, the logical connectives simply act like operators on truth values. They do not try to judge relevance, causality, etc.

The logical connectives' behavior can be summarized by **truth tables**.

Negation ($\neg$, "not", "it is not true that")

Meaning: "It is not true that ______."

P	$\neg P$
T	F
F	T

Examples

- "I don't like pears" $=$ $\neg$("I like pears") $=$ $\neg$F $=$ T
- "today is not Monday" $=$ $\neg$("today is Monday") $=$ $\neg$T $=$ F

(This lecture is delivered on a Monday.)

Conjunction ($\wedge$, "and")

Meaning: "________ and ________."

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples

- "$2 < 3$ and $3 < 5$" $=$ $(2 < 3) \wedge (3 < 5)$ $=$ T $\wedge$ T $=$ T
- "it is cloudy, and I have an umbrella"
  $=$ "it is cloudy" $\wedge$ "I have an umbrella" $=$ T $\wedge$ F $=$ F

Disjunction ($\lor$, "or")

Meaning: "________ or ________ (or both)."

Note: Disjunction is "inclusive or" — it allows both cases to be true. That is often not what "or" means in ordinary English. ("You can have soup *or* salad.")

P	Q	$P \lor Q$
T	T	T
T	F	T
F	T	T
F	F	F

Examples

- "it is Monday or the moon is made of cheese"
  $=$ "it is Monday" $\lor$ "moon is cheese" $=$ T $\lor$ F $=$ T
- "72 is a multiple of 6 or 8"
  $=$ "72 is a multiple of 6" $\lor$ "72 is a multiple of 8" $=$ T $\lor$ T $=$ T

Exclusive Or ($\oplus$, XOR)

Meaning: "________ or ________ (but not both)."

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Not used in mathematical logic, but common in programming, circuits, etc.

$P \oplus Q$ is equivalent to $(P \lor Q) \land \neg(P \land Q)$.

$P \oplus Q$ is also equivalent to $(P \land \neg Q) \lor (\neg P \land Q)$.

$P \oplus Q$ is also equivalent to $\neg(P \Leftrightarrow Q)$.

Implication ($\Rightarrow$, "if–then", "implies")

Meaning: "If ________, then ________."

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P is called the **hypothesis**; Q is called the **conclusion**.

$P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

Examples

- "if today is Monday, then you have class" = T $\Rightarrow$ T = T
- "if $1 = 2$, then I am the king of France" = F $\Rightarrow$ F = T

Biconditional ("if and only if", sometimes written "iff")

Meaning: "_______ if and only if _______."

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \land (Q \Rightarrow P)$.

$P \Leftrightarrow Q$ is also equivalent to $(P \land Q) \lor (\neg P \land \neg Q)$.

Summary: Logical Connectives

P	$\neg P$
T	F
F	T

P	Q	$P \land Q$	$P \lor Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Evaluating Propositions

Truth tables can be used to evaluate complex propositions.

1. Create a column for each propositional variable, and create $2^{\text{\#vars}}$ rows. Fill in every combination of truth values for the propositional variables.
2. Create a column for each sub-expression, smallest first.
3. Fill in each column by applying the truth table of its main connective to the truth values of its arguments *from that row*.

Example

Evaluate the proposition: $\neg P \vee \neg Q$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Evaluating Multiple Propositions

You can evaluate multiple statements in a truth table.

Example

Evaluate $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$.

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P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Evaluating Multiple Propositions

You can evaluate multiple statements in a truth table.

Example

Evaluate $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Notice that the $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ columns have the same values. The two propositions are **logically equivalent**.

Logically Equivalent Statements

If two propositions X and Y are logically equivalent,
then the proposition $X \Leftrightarrow Y$ always evaluates to T, and vice versa.

Example

P	Q	$\neg(P \land Q)$	$\neg P \lor \neg Q$	$\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

Exercise: Implications and Logical Equivalence

Make a truth table with the following four propositions:

$$P \Rightarrow Q \qquad Q \Rightarrow P \qquad \neg P \Rightarrow \neg Q \qquad \neg Q \Rightarrow \neg P$$

Are any of those propositions logically equivalent?

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Are any of those propositions logically equivalent?

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Exercise: Implications and Logical Equivalence

Make a truth table with the following four propositions:

$$P \Rightarrow Q \qquad Q \Rightarrow P \qquad \neg P \Rightarrow \neg Q \qquad \neg Q \Rightarrow \neg P$$

Are any of those propositions logically equivalent?

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

$P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$, and
$Q \Rightarrow P$ is equivalent to $\neg P \Rightarrow \neg Q$.

Contrapositive

Definition (Contrapositive)

The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

It is logically equivalent to the original proposition.

Examples

- **Original:** "If today is Tuesday, then you have class."
  Contrapositive: "If you don't have class, then today is not Tuesday."
- **Original:** "If my light is green, then the other light is red."
  Contrapositive: "If the other light is not red, then my light is not green."

Logical Equivalences

Name	Disjunction	Conjunction
Identity	$A \vee \mathsf{F} \;\Leftrightarrow\; A$	$A \wedge \mathsf{T} \;\Leftrightarrow\; A$
Dominance	$A \vee \mathsf{T} \;\Leftrightarrow\; \mathsf{T}$	$A \wedge \mathsf{F} \;\Leftrightarrow\; \mathsf{F}$
Idempotent	$A \vee A \;\Leftrightarrow\; A$	$A \wedge A \;\Leftrightarrow\; A$
Inverse	$A \vee \neg A \;\Leftrightarrow\; \mathsf{T}$	$A \wedge \neg A \;\Leftrightarrow\; \mathsf{F}$
Commutative	$A \vee B \;\Leftrightarrow\; B \vee A$	$A \wedge B \;\Leftrightarrow\; B \wedge A$
Associative	$(A \vee B) \vee C \;\Leftrightarrow\; A \vee (B \vee C)$	$(A \wedge B) \wedge C \;\Leftrightarrow\; A \wedge (B \wedge C)$
Distributive	$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
Absorption	$A \vee (A \wedge B) \;\Leftrightarrow\; A$	$A \wedge (A \vee B) \;\Leftrightarrow\; A$
DeMorgan	$\neg(A \vee B) \;\Leftrightarrow\; \neg A \wedge \neg B$	$\neg(A \wedge B) \;\Leftrightarrow\; \neg A \vee \neg B$

Logical Equivalences

Name	Equivalence
Double Negation	$\neg\neg A \;\Leftrightarrow\; A$
Conditional	$A \Rightarrow B \;\Leftrightarrow\; \neg A \vee B$
Contrapositive	$A \Rightarrow B \;\Leftrightarrow\; \neg B \Rightarrow \neg A$
Biconditional	$(A \Leftrightarrow B) \;\Leftrightarrow\; ((A \Rightarrow B) \wedge (B \Rightarrow A))$

Main Connective

A complex proposition may contain many logical connectives.

The **main connective** is the one that forms the whole proposition.
That is, the main connective is not inside of any sub-proposition.

Examples

What is the main connective of each of the following propositions?

- $P \land \neg Q$
- $(A \Rightarrow B) \Rightarrow (B \lor \neg A)$
- $X \lor (Y \land Z) \lor (\neg Y \land \neg Z)$
- $\neg(P \Rightarrow (Q \land R))$

Tautologies and Contradictions

Definitions (Tautology, Contradiction)

A **tautology** is a proposition that is always true.

A **contradiction** is a proposition that is always false.

A **contingent proposition** is neither always true nor always false.
Its truth value depends on the truth values of its propositional variables.

Examples (Tautologies)

- $R \lor \neg R$
- $\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

Examples (Contradictions)

- $R \land \neg R$
- $(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$

The negation of any tautology is a contradiction.
The negation of any contradiction is a tautology.
The negation of any contingent proposition is contingent.

Satisfiability

Definition (Truth Assignment)

A **truth assignment** maps propositional variables to truth values.
Each row of a truth table corresponds to a truth assignment.

Definition (Satisfiable)

A proposition is **satisfiable** if there is a truth assignment that makes it true.
A proposition is **valid** if every truth assignment makes it true.
A proposition is **unsatisfiable** if every truth assignment makes it false.
(That is, valid = tautology, unsatisfiable = contradiction.)

Is there an algorithm for determining if a proposition is satisfiable?
Is there an algorithm for determining if a proposition is valid?

From Truth Table to Proposition

Given a truth table, can we find a proposition that has that truth table?

Example

P	Q	R	?
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
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F	T	F	F
F	F	T	T
F	F	F	T

We can get a true result by:

picking row 4 OR
picking row 7 OR
picking row 8

Under what circumstances does row 4 apply? (Likewise, 7 and 8.)

From Truth Table to Proposition

Given a truth table, can we find a proposition that has that truth table?

Example

P	Q	R	?	
T	T	T	F	
T	T	F	F	
T	F	T	F	
T	F	F	T	$P \wedge \neg Q \wedge \neg R$
F	T	T	F	
F	T	F	F	
F	F	T	T	$\neg P \wedge \neg Q \wedge R$
F	F	F	T	$\neg P \wedge \neg Q \wedge \neg R$

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Example

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T	T	T	F	
T	T	F	F	
T	F	T	F	
T	F	F	T	$P \wedge \neg Q \wedge \neg R$
F	T	T	F	
F	T	F	F	
F	F	T	T	$\neg P \wedge \neg Q \wedge R$
F	F	F	T	$\neg P \wedge \neg Q \wedge \neg R$

We can get a true result by:

picking row 4 OR
picking row 7 OR
picking row 8

Under what circumstances does row 4 apply?
(Likewise, 7 and 8.)

Solution:

$$(P \wedge \neg Q \wedge \neg R) \quad \vee \quad (\neg P \wedge \neg Q \wedge R) \quad \vee \quad (\neg P \wedge \neg Q \wedge \neg R)$$

This proposition is in disjunctive normal form.

Rewriting with Logical Equivalences

That technique gives us *some* proposition. Is it the best? The shortest?

We can rewrite the proposition using logical equivalences:

$$
\begin{aligned}
& (P \land \neg Q \land \neg R) \;\lor\; (\neg P \land \neg Q \land R) \;\lor\; (\neg P \land \neg Q \land \neg R) && \\
=\;& \neg Q \;\land\; [(P \land \neg R) \;\lor\; (\neg P \land R) \;\lor\; (\neg P \land \neg R)] && \text{Distrib.} \\
=\;& \neg Q \;\land\; [(P \land \neg R) \;\lor\; (\neg P \land \neg R) \;\lor\; (\neg P \land R)] && \text{Commut.} \\
=\;& \neg Q \;\land\; [(P \land \neg R) \;\lor\; (\neg P \land \neg R) \;\lor\; (\neg P \land \neg R) \;\lor\; (\neg P \land R)] && \text{Idem.} \\
=\;& \neg Q \;\land\; [((P \lor \neg P) \land \neg R) \;\lor\; (\neg P \land (\neg R \lor R))] && \text{Distrib.} \\
=\;& \neg Q \;\land\; [(\mathsf{T} \land \neg R) \;\lor\; (\neg P \land \mathsf{T})] && \text{Inv.} \\
=\;& \neg Q \;\land\; [\neg R \;\lor\; \neg P] && \text{Ident.} \\
=\;& \neg Q \;\land\; \neg (R \land P) && \text{DeMorgan} \\
=\;& \neg (Q \lor (R \land P)) && \text{DeMorgan}
\end{aligned}
$$

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We can rewrite the proposition using logical equivalences:

$$
\begin{array}{rll}
& (P \land \neg Q \land \neg R) \ \lor \ (\neg P \land \neg Q \land R) \ \lor \ (\neg P \land \neg Q \land \neg R) & \\
= & \neg Q \ \land \ [(P \land \neg R) \ \lor \ (\neg P \land R) \ \lor \ (\neg P \land \neg R)] & \text{Distrib.} \\
= & \neg Q \ \land \ [(P \land \neg R) \ \lor \ (\neg P \land \neg R) \ \lor \ (\neg P \land R)] & \text{Commut.} \\
= & \neg Q \ \land \ [(P \land \neg R) \ \lor \ (\neg P \land \neg R) \ \lor \ (\neg P \land \neg R) \ \lor \ (\neg P \land R)] & \text{Idem.} \\
= & \neg Q \ \land \ [((P \lor \neg P) \land \neg R) \ \lor \ (\neg P \land (\neg R \lor R))] & \text{Distrib.} \\
= & \neg Q \ \land \ [(\mathsf{T} \land \neg R) \ \lor \ (\neg P \land \mathsf{T})] & \text{Inv.} \\
= & \neg Q \ \land \ [\neg R \ \lor \ \neg P] & \text{Ident.} \\
= & \neg Q \ \land \ \neg(R \land P) & \text{DeMorgan} \\
= & \neg(Q \lor (R \land P)) & \text{DeMorgan}
\end{array}
$$

This should remind you of algebra.

High-school algebra is mainly about $\mathbb{R}$ with the operations $+$ and $\cdot$ (and $-$ and $\div$).
The algebra of $\{\mathsf{T}, \mathsf{F}\}$ with the operations $\lor, \land, \neg$ is called Boolean algebra.

Topic List

- statement, proposition
- domain knowledge vs logical structure
- propositional logic: variables, compound propositions, connectives
- truth value, truth tables
- connectives: not ($\neg$), and ($\wedge$), or ($\vee$), conditional/implies ($\Rightarrow$), iff/biconditional ($\Leftrightarrow$)
- evaluating the truth of propositions
- logical equivalence, logical equivalences
- tautology, contradiction, contingent
- satisfiable, valid, unsatisfiable
- from truth table to proposition in DNF