Predicate Logic

CS 220 — Applied Discrete Mathematics

February {5, 10}, 2025

Back to Mathematics

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If $x > 10$, *then* $x > 5$.

Can we fit this into our logical framework?

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Let A represent "$x > 10$" and let B represent "$x > 5$".

Then $A \Rightarrow B$.

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There's a relevant general rule:

If $x > y$ and $y > z$,
then $x > z$.

Can we express that rule in logic?

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Let P represent "$x > y$" and let Q represent "$y > z$" and let R represent "$x > z$".

Then $(P \wedge Q) \Rightarrow R$.

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Can we express that rule in logic?

Let P represent "$x > y$"
and let Q represent "$y > z$"
and let R represent "$x > z$".

Then $(P \land Q) \Rightarrow R$.

Can we use the **rule** $(P \land Q) \Rightarrow R$ to show the **instance** $A \Rightarrow B$?

No. Propositional logic is insufficient.

We need a logic that can talk about *things* and *relationships between things*.

Open Statements

Definition (Open statement)

An **open statement** is a sentence that contains **object variables** whose values are not known. If the variables were replaced with specific objects, the sentence would be a statement.

Examples

- The arrays A and B have the same length.
- The priority of every job in `ready_queue` is less than p_{max}.

We sometimes view an open statement as a statement-valued function.

Example

- $E(k)$ = "The integer k is even." (open statement)
- $E(7)$ = "The integer 7 is even." (statement, false)

Predicate Logic

Definition (Predicate Logic)

Predicate logic (aka **first-order logic**) is an extension of propositional logic that can also talk about objects and predicates about objects.

A proposition in predicate logic is one of the following:

- a propositional variable
- a compound proposition formed by a logical connective
- a predicate name P applied to one or more object expressions
- a quantified proposition

An **object expression** is one of the following:

- a literal object, like 12, Boston, or $\{1, 3, 5\}$
- an object variable that is in scope
- applications of functions, operators, etc to other object expressions

Predicates

Examples (Object Expressions)

- 12
- 3 + 4
- Boston
- $\{1, 4, 9\}$
- x — x must be in scope
- $1 + \sin(\pi k)$

Examples (Propositions using Predicates)

- $\text{Likes}($Jenny, Back to the Future$)$ — Likes is a predicate name
- $\text{In}(city,$ MA$)$ — In is a predicate name
- $x = 5$
- $n \in \mathbb{N}$
- $5 \in A$

Jenny, Back to the Future, and MA are not object variables; they are literal objects (aka **constants**), like 5, $\mathbb{N}$.
Predicates named by symbols, like "$=$", "$\in$", and "$\subseteq$", are usually written between their arguments.

Quantified Propositions

Quantifier	Proposition	Read as
Universal	$\forall x \in S,\ P$	"for **all** x in S, P"
Existential	$\exists x \in S,\ P$	"there **exists** x in S such that P"

The **quantifier body** P can be any proposition; x is **in scope** inside of P.

Examples

- "**Every** real number is less than, equal to, or greater than zero."
  $\forall x \in \mathbb{R},\ (x < 0 \ \lor\ x = 0 \ \lor\ x > 0)$
- "**For every** real number, **there is** a smaller real number."
  $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y < x$
- "**There is** a smallest natural number."
  $\exists z \in \mathbb{N},\ \forall n \in \mathbb{N},\ z \leq n$
- "**There is** a color other than blue."
  $\exists c \in \mathit{Color},\ c \neq$ blue

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  $\exists c \in \mathit{Color},\ \boxed{c \neq \text{blue}}$

Univeral Quantifier as Conjunction

The universal quantifier acts like a (possibly infinite) conjunction:

$$\forall n \in \mathbb{N},\ P(n) = \bigwedge_{n \in \mathbb{N}} P(n) = P(0) \land P(1) \land P(2) \land \ldots$$

The proposition is true when P holds **for every** element of the given set.

$$\forall x \in S,\ P(x) = \bigwedge_{x \in S} P(x) =$$

```
for x ∈ S do
  if P(x) then
    continue
  else
    return F
  end if
end for
return T
```

A value of x that makes $P(x)$ **false** is called a **counterexample**.

Existential Quantifier as Disjunction

The existential quantifier acts like a (possibly infinite) disjunction:

$$\exists n \in \mathbb{N}, P(n) = \bigvee_{n \in \mathbb{N}} P(n) = P(0) \vee P(1) \vee P(2) \vee \ldots$$

The proposition is true when P holds **for some** (at least one, maybe more) element of the given set.

$$\exists x \in S, P(x) \; = \; \bigvee_{x \in S} P(x) \; =$$

```
for x ∈ S do
  if P(x) then
    return T
  else
    continue
  end if
end for
return F
```

A value of x that makes $P(x)$ **true** is called a **witness**.

Exercises: Evaluating Quantified Propositions

Let $H = \{1, 2, 3, 4, 5\}$. Judge the truth of the following propositions. Provide a witness or counterexample if appropriate.

- $\exists n \in H,\ \mathrm{Odd}(n)$
- $\forall n \in H,\ \mathrm{Even}(n)$
- $\forall n \in H,\ \exists m \in H,\ m + n = 6$
- $\exists n \in H,\ \forall m \in H,\ \mathrm{Odd}(m) \Rightarrow m < n$
- $\exists n \in H,\ \forall m \in H,\ \mathrm{Even}(m) \Rightarrow m < n$

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- $\exists n \in H,\ \mathrm{Odd}(n)$

  T with witness $n = 1$ (or 3 or 5)
- $\forall n \in H,\ \mathrm{Even}(n)$
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  T, each n has an m-witness:

n	1	2	3	4	5
m	5	4	3	2	1

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n	1	2	3	4	5
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- $\exists n \in H,\ \forall m \in H,\ \mathrm{Odd}(m) \Rightarrow m < n$

  F, each n has an m-counterexample:

n	1	2	3	4	5
m	5	5	5	5	5

  (etc)
- $\exists n \in H,\ \forall m \in H,\ \mathrm{Even}(m) \Rightarrow m < n$

Exercises: Evaluating Quantified Propositions

Let $H = \{1, 2, 3, 4, 5\}$. Judge the truth of the following propositions. Provide a witness or counterexample if appropriate.

- $\exists n \in H,\ \mathrm{Odd}(n)$

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- $\forall n \in H,\ \exists m \in H,\ m + n = 6$

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n	1	2	3	4	5
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  (etc)

- $\exists n \in H,\ \forall m \in H,\ \mathrm{Even}(m) \Rightarrow m < n$

  T with witness $n = 5$

DeMorgan's Laws for Quantifiers

$$\neg(A \land B) = \neg A \lor \neg B \qquad \neg(\forall x \in S,\ P(x)) = \exists x \in S,\ \neg P(x)$$

$$\neg(A \lor B) = \neg A \land \neg B \qquad \neg(\exists x \in S,\ P(x)) = \forall x \in S,\ \neg P(x)$$

Examples

- "It is not true that every natural number is even."
  = "There exists some natural number that is not even."
- "It is not true that there exists some person who is immortal."
  = "Every person is not immortal." = "Every person is mortal."
- "It is not true that there is someone who is both garrulous and taciturn."
  = "For every person, they are not both garrulous and taciturn."
  = "For every person, either they are not garrulous or they are not taciturn."

Translating Statements into Predicate Logic

Translating English statements into predicate logic propositions:

- **Rephrase** the conversational English into the pseudo-English constructs below, working from the "outside" inward. Watch out for
  - implicitly universal statements
  - verbs with compound objects (eg, "I like apples and oranges.")
  - verbs with quantified objects (eg, "10 is greater than some odd number.")
- **Replace** pseudo-English with logical quantifiers and connectives. **Replace** simple statements with uses of predicates.

Pseudo-English	**Logic**
for every $Type\ x$, ____	$\forall x \in Type,$ ____
there exists some $Type\ x$ such that ____	$\exists x \in Type,$ ____
____ and ____	____ $\wedge$ ____
____ or ____	____ $\vee$ ____
if ____ then ____	____ $\Rightarrow$ ____
it is not true that ____	$\neg($____$)$

Examples: Translating Statements

Sets:

$$P = \text{a set of people} \qquad A = \text{a set of actors}$$
$$M = \text{a set of movies} \qquad G = \text{a set of genres}$$

Predicates:

$$\begin{aligned} \text{Likes}(p, x) &\quad \text{where } p \in P,\ x \in (M \cup G \cup A) \\ \text{HasGenre}(m, g) &\quad \text{where } m \in M,\ g \in G \\ \text{ActedIn}(a, m) &\quad \text{where } a \in A,\ m \in M \end{aligned}$$

Examples: Translating Statements

1. "Everyone has some movie that they like."

2. "There is some movie that everyone likes."

3. "Every movie has some fan (a person who likes it)."

Examples: Translating Statements

1. "Everyone has some movie that they like."
   = "for every person p, p has some movie that they like"
   = $\forall p \in P,$ "p has some movie that they like"
   = $\forall p \in P,$ "there exists some movie m such that p likes m"
   = $\forall p \in P,\ \exists m \in M,$ "p likes m"
   = $\forall p \in P,\ \exists m \in M,\ \mathrm{Likes}(p, m)$
2. "There is some movie that everyone likes."

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   = $\forall p \in P$, "p has some movie that they like"
   = $\forall p \in P$, "there exists some movie m such that p likes m"
   = $\forall p \in P,\ \exists m \in M$, "$p$ likes m"
   = $\forall p \in P,\ \exists m \in M,\ \text{Likes}(p, m)$
2. "There is some movie that everyone likes."
   = "there exists a movie m such that everyone likes m"
   = $\exists m \in M$, "everyone likes m"
   = $\exists m \in M$, "for every person p, p likes m"
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3. "Every movie has some fan (a person who likes it)."
   = "for every movie m, m has some fan (a person who likes it)"
   = $\forall m \in M$, "m has some fan (a person who likes it)"
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   = $\forall m \in M,\ \exists p \in P,\ \mathrm{Likes}(p, m)$

Examples: Translating Statements

Summary:

1. "Everyone has some movie that they like."
   = $\forall p \in P,\ \exists m \in M,\ \mathrm{Likes}(p, m)$
2. "There is some movie that everyone likes."
   = $\exists m \in M,\ \forall p \in P,\ \mathrm{Likes}(p, m)$
3. "Every movie has some fan (a person who likes it)."
   = $\forall m \in M,\ \exists p \in P,\ \mathrm{Likes}(p, m)$

Note:

- Quantifier order matters! Compare #1 ($\forall p, \exists m$) and #2 ($\exists m, \forall p$).
- Quantifier choice matters! Compare #2 ($\exists m, \forall p$) and #3 ($\forall m, \exists p$).

Exercise: Translating Statements

$\forall x \in Type,\ P$ = "for every $Type\ x$, P"
$\exists x \in Type,\ P$ = "there exists some $Type\ x$ such that P"

1. Everyone likes Pulp Fiction or Bridesmaids.
2. No one likes Borderlands.
3. If a person likes The Matrix, they also like John Wick.
4. If someone likes Batman & Robin, they like every movie.
5. Everyone likes someone who likes Sharknado.

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   $\neg\exists p \in P,\ \text{Likes}(p, \text{Borderlands})$
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   $\forall p \in P,\ \text{Likes}(p, \text{Batman \& Robin}) \Rightarrow (\forall m \in M,\ \text{Likes}(p, m))$
5. Everyone likes someone who likes Sharknado. (ambiguous)
   $\forall p \in P,\ \exists q \in P,\ \text{Likes}(p, q) \land \text{Likes}(q, \text{Sharknado})$ or
   $\forall p \in P,\ \forall q \in P,\ \text{Likes}(q, \text{Sharknado}) \Rightarrow \text{Likes}(p, q)$

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Everyone needs someone with a minivan.
Everyone avoids someone with a nasty cough.

Quantifiers and Sets

Recall the definition of subset:

Definition (Subset)

Suppose A and B are sets. A is a **subset** of B, written $A \subseteq B$, if every element of A is also an element of B.

Can we express this definition in predicate logic?

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Can we express this definition in predicate logic?

Suppose A *and* B *are sets. Then* $A \subseteq B \iff \forall x \in A,\ x \in B.$

But A and B are variables. Shouldn't they be bound by quantifiers?

$$\forall A, B \in ???,\ \ (A \subseteq B \iff \forall x \in A,\ x \in B)$$

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But A and B are variables. Shouldn't they be bound by quantifiers?

$$\forall A, B \in ???,\ \ (A \subseteq B \iff \forall x \in A,\ x \in B)$$

There is no "set of all sets".* I'll use the following notation instead:

$$\forall A, B : \textbf{\textit{Set}},\ \ (A \subseteq B \iff \forall x \in A,\ x \in B)$$

Restrictions on Set

Do not use **Set** inside of set-builder notation.

Some Definitions Restated

$\forall A, B : \textbf{Set},\ (A \subseteq B \iff \forall x \in A,\ x \in B)$	Subset
$\forall A, B : \textbf{Set},\ (A \subset B \iff A \subseteq B \land A \neq B)$	Proper Subset
$\forall A, B : \textbf{Set},\ A \times B = \{(a, b) \mid a \in A,\ b \in B\}$	Cartesian Product
$\forall A : \textbf{Set},\ \mathscr{P}(A) = \{S \mid S \subseteq A\}$	Power Set
$\forall A, B : \textbf{Set},\ A \cup B = \{x \mid x \in A \lor x \in B\}$	Union
$\forall A, B : \textbf{Set},\ A \cap B = \{x \mid x \in A \land x \in B\}$	Intersection
$\forall A, B : \textbf{Set},\ A - B = \{x \mid x \in A \land x \notin B\}$	Difference

Quantifiers without Set Bounds

Some versions of predicate logic use quantifiers where the variables does not have an associated set:

$$\forall x,\ P(x) \qquad\qquad \exists x,\ P(x)$$

Then the variable ranges over the implicit **universe of discourse**.

You can convert between bounded and unbounded quantifiers, but each quantifier has a different conversion rules:

$$\begin{aligned} \forall x \in S,\ P(x) &= \forall x,\ x \in S \Rightarrow P(x) \\ \exists x \in S,\ P(x) &= \exists x,\ x \in S \wedge P(x) \end{aligned}$$

Quantifiers in the textbooks

Applied Discrete Mathematics uses quantifiers without set bounds.
Book of Proof uses quantifiers with set bounds.

Topic List

- open statement
- predicate logic: predicates, object variables
- quantifiers: universal ($\forall$) vs existential ($\exists$)
- evaluating quantified propositions
- translating statements into predicate logic