Relations

CS 220 — Applied Discrete Mathematics

February {12, 19, 24}, 2025

Predicate Logic

Let P be a set of people. Here are some predicates on P:

$$
\begin{aligned}
\text{SiblingOf}(a,b) &= \text{“}a \text{ and } b \text{ are siblings”} && \text{where } a,b \in P \\
\text{ChildOf}(c,p) &= \text{“}c \text{ is a child of } p\text{”} && \text{where } c,p \in P \\
\text{DescendantOf}(a,d) &= \text{“}d \text{ is a descendant of } a\text{”} && \text{where } a,d \in P \\
\text{RelatedTo}(a,b) &= \text{“}a \text{ and } b \text{ are related (or equal)”} && \text{where } a,b \in P
\end{aligned}
$$

Interesting facts about these predicates:

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$$\text{RelatedTo}(a,b) = \text{“}a \text{ and } b \text{ are related (or equal)”} \qquad \text{where } a,b \in P$$

Interesting facts about these predicates:

- (Symmetry) If $\text{SiblingOf}(a,b)$, then $\text{SiblingOf}(b,a)$.
- (Transitivity) If $\text{DescendantOf}(a,b)$ and $\text{DescendantOf}(b,c)$, then $\text{DescendantOf}(a,c)$.
- RelatedTo has both symmetry and transitivity.
- If you have ChildOf, you could generate DescendantOf. (How?)

Predicate Logic

numbers	are objects
sets	are objects
predicates	represent properties of objects
	represent relationships between objects
???	represents properties of predicates

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(We can't define predicates or functions on predicates.)

Predicate Logic

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	represent relationships between objects
???	represents properties of predicates

In **first-order** predicate logic, predicates are not objects.
(We can't define predicates or functions on predicates.)

Solutions:

- higher-order logic (not in this class; see CS 420, CS 720)
- find a way to represent properties and relationships *as objects*

Representing Properties and Relationships

Represent properties and relationships as objects — specifically, sets.

- Represent a property of elements of A as a subset of A.
  Examples:
- Represent a relationship on elements of A as a subset of $A \times A$.
  Examples:
- Represent a relationship between A and B as a subset of $A \times B$.
  Examples:
- ...

Representing Properties and Relationships

Represent properties and relationships as objects — specifically, sets.

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  Examples: $Even \subseteq \mathbb{Z}$, $Prime \subseteq \mathbb{N}$
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- Represent a relationship on elements of A as a subset of $A \times A$.
  Examples: $(<) \subseteq \mathbb{R} \times \mathbb{R}$, $\mathit{SiblingOf} \subseteq \mathit{Person} \times \mathit{Person}$
- Represent a relationship between A and B as a subset of $A \times B$.
  Examples:
- ...

Representing Properties and Relationships

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- Represent a property of elements of A as a subset of A.
  Examples: $\mathit{Even} \subseteq \mathbb{Z}$, $\mathit{Prime} \subseteq \mathbb{N}$
- Represent a relationship on elements of A as a subset of $A \times A$.
  Examples: $(<) \subseteq \mathbb{R} \times \mathbb{R}$, $\mathit{SiblingOf} \subseteq \mathit{Person} \times \mathit{Person}$
- Represent a relationship between A and B as a subset of $A \times B$.
  Examples: $\mathit{ActedIn} \subseteq \mathit{Actors} \times \mathit{Movies}$, $\mathit{HasLivedIn} \subseteq \mathit{Person} \times \mathit{City}$
- ...

Relations

Definition (Binary Relation)

Let A and B be sets.

- A **binary relation on A** is a subset of $A \times A$.
- A **binary relation from A to B** is a subset of $A \times B$.

We often drop the qualifier “binary”, but there are other kinds of relations.

We use relations to model relationships between things.

Examples

Example relations on $Person$:

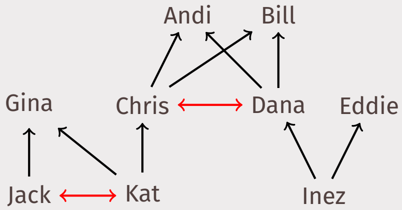
- is a child of
- is friends with
- is older than

Example relations from $Person$ to $City$:

- lives in
- was born in
- has visited

Example: Relations on People

$$\mathit{ChildOf}, \mathit{SiblingOf} \subseteq \mathit{People} \times \mathit{People}$$



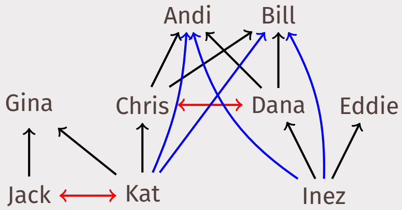
$$\begin{aligned}\mathit{ChildOf} = \{&(\text{Chris}, \text{Andi}), (\text{Chris}, \text{Bill}),\\ &(\text{Dana}, \text{Andi}), (\text{Dana}, \text{Bill}),\\ &(\text{Inez}, \text{Dana}), (\text{Inez}, \text{Eddie}),\\ &(\text{Jack}, \text{Gina}),\\ &(\text{Kat}, \text{Gina}), (\text{Kat}, \text{Chris})\}\end{aligned}$$

$$\begin{aligned}\mathit{SiblingOf} = \{&(\text{Chris}, \text{Dana}), (\text{Dana}, \text{Chris}),\\ &(\text{Jack}, \text{Kat}), (\text{Kat}, \text{Jack})\}\end{aligned}$$

(relations drawn as a directed graph, aka digraph)

Example: Relations on People

$$\mathit{ChildOf}, \mathit{SiblingOf} \subseteq \mathit{People} \times \mathit{People}$$



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$$\begin{aligned}
\mathit{ChildOf} = \{ & (\text{Chris}, \text{Andi}), (\text{Chris}, \text{Bill}),\\
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& (\text{Inez}, \text{Dana}), (\text{Inez}, \text{Eddie}),\\
& (\text{Jack}, \text{Gina}),\\
& (\text{Kat}, \text{Gina}), (\text{Kat}, \text{Chris})\}
\end{aligned}$$

$$\begin{aligned}
\mathit{SiblingOf} = \{ & (\text{Chris}, \text{Dana}), (\text{Dana}, \text{Chris}),\\
& (\text{Jack}, \text{Kat}), (\text{Kat}, \text{Jack})\}
\end{aligned}$$

$$\begin{aligned}
\mathit{DescendantOf} = \{ & (\text{Inez}, \text{Andi}), (\text{Inez}, \text{Bill}),\\
& (\text{Kat}, \text{Andi}), (\text{Kat}, \text{Bill})\}\\
& \cup \mathit{ChildOf}
\end{aligned}$$

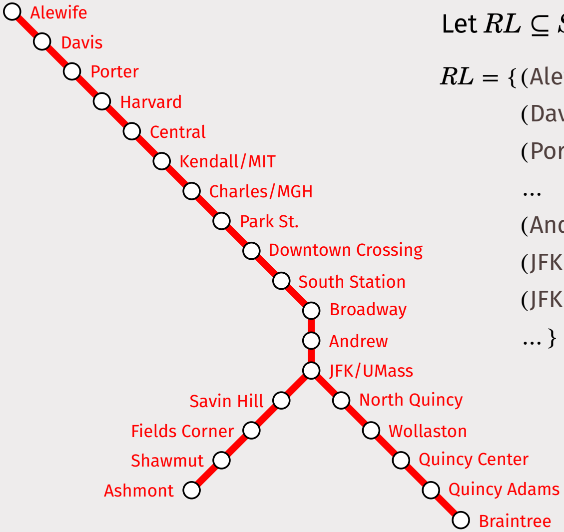
Example: Relations on Numbers

Let $H = \{1, 2, 3, 4, 5\}$, and let S, LT, and LE be defined as follows:

$$\begin{aligned}
S &= \{(1,2), (2,3), (3,4), (4,5)\} \\
LT &= \{(1,2), (1,3), (1,4), (1,5), \\
&\quad\ (2,3), (2,4), (2,5), \\
&\quad\ (3,4), (3,5), \\
&\quad\ (4,5)\} \\
LE &= \{(1,1), (2,2), (3,3), (4,4), (5,5)\} \cup LT
\end{aligned}$$

S represents the “successor” relation on H.
LT represents the “less than” relation on H.
LE represents “less than or equal to” relation on H.

Example: Relation on T Stations



Let $St = \{\text{Alewife}, \text{Davis}, \ldots\}$

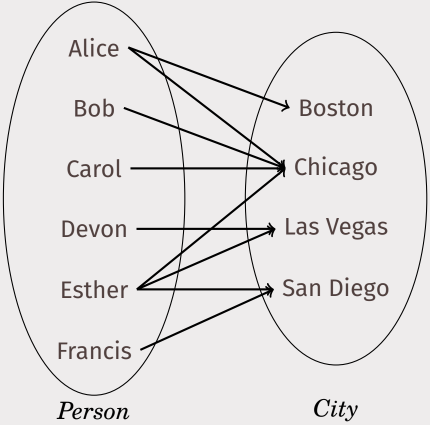
Let $RL \subseteq St \times St$ be defined as follows:

$$
\begin{aligned}
RL = \{ & (\text{Alewife}, \text{Davis}), \\
& (\text{Davis}, \text{Porter}), (\text{Davis}, \text{Alewife}), \\
& (\text{Porter}, \text{Davis}), (\text{Porter}, \text{Harvard}), \\
& \ldots \\
& (\text{Andrew}, \text{JFK/UMass}), (\text{Andrew}, \text{Broadway}), \\
& (\text{JFK/UMass}, \text{Andrew}), (\text{JFK/UMass}, \text{Savin Hill}), \\
& (\text{JFK/UMass}, \text{North Quincy}), \\
& \ldots \}
\end{aligned}
$$

$(s_1, s_2) \in RL$ means
"can get from s_1 to s_2
in one step on Red Line"

Examples: Relation from Person to City

$$HasLivedIn \subseteq Person \times City$$



$HasLivedIn$ =

{(Alice, Boston), (Alice, Chicago),
(Bob, Chicago),
(Carol, Chicago),
(Devon, Las Vegas),
(Esther, Chicago), (Esther, Las Vegas),
(Esther, San Diego),
(Francis, San Diego)}

Relations vs Predicates

Relations and predicates serve the same purpose: to represent relationships between things. Keep the formal distinction in mind:

- A relation is a set. It may contain some tuples and not others.
- A predicate is used in an open proposition, which may be true for some values of object variables and false for others.

A relation can be defined by an open proposition using set-builder notation:

$$\underbrace{Divides = \{(d,n) \mid d \in \mathbb{N},\ n \in \mathbb{N},\ \underbrace{\exists k \in \mathbb{N},\ kd = n}_{\text{open proposition on } d,n}\}}_{\text{relation}}$$

For example, $(3, 27) \in Divides$ because $\exists k \in \mathbb{N},\ 3k = 27$. ($k = 9$ works.)

Notation for Relations

Some relations are named by symbols.

If R is a binary relation, it is common to write xRy instead of $(x, y) \in R$.

Example

$<$ is a relation on $\mathbb{R}$
We typically write $3 < 5$ instead of $(3, 5) \in <$.

Properties of Relations

Reflexivity

Definition (Reflexive)

A relation R on a set A is **reflexive** iff $(a, a) \in R$ for every $a \in A$.

That is, every element is related to itself.

Examples (Reflexive)

Let $S = \{1, 2, 3, 4\}$. Which of the following relations on S is reflexive?

1. $\{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$
2. $\{(1, 1), (2, 2), (2, 3), (3, 3), (4, 1), (4, 4)\}$
3. $\{(1, 1), (2, 2), (3, 3)\}$

Definition (Irreflexive)

A relation R on A is **irreflexive** if $(a, a) \notin R$ for all $a \in R$.

That is, no element is related to itself.

Reflexivity

Definition (Reflexive)

A relation R on a set A is **reflexive** iff $(a, a) \in R$ for every $a \in A$.

That is, every element is related to itself.

Examples (Reflexive)

Let $S = \{1, 2, 3, 4\}$. Which of the following relations on S is reflexive?

1. $\{(1,1), (1,2), (2,3), (3,3), (4,4)\}$ — no, missing $(2,2)$
2. $\{(1,1), (2,2), (2,3), (3,3), (4,1), (4,4)\}$ — yes
3. $\{(1,1), (2,2), (3,3)\}$ — no*, missing $(4,4)$

Definition (Irreflexive)

A relation R on A is **irreflexive** if $(a, a) \notin R$ for all $a \in R$.

That is, no element is related to itself.

Symmetry

Definition (Symmetric)

A relation R on a set A is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$.

Examples (Symmetric)

Let $S = \{1, 2, 3, 4\}$. Which of the following relations is symmetric?

1. $\{(1,1), (1,2), (1,3), (2,1), (3,1)\}$
2. $\{(1,2), (2,3), (3,4)\}$
3. $\{(2,2), (3,3)\}$

Definitions (Antisymmetric, Asymmetric)

A relation R on A is **antisymmetric** iff whenever $(a, b) \in R$ and $(b, a) \in R$, $a = b$.

A relation R on A is **asymmetric** iff $(a, b) \in R$ implies that $(b, a) \notin R$.

Symmetry

Definition (Symmetric)

A relation R on a set A is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$.

Examples (Symmetric)

Let $S = \{1, 2, 3, 4\}$. Which of the following relations is symmetric?

1. $\{(1,1), (1,2), (1,3), (2,1), (3,1)\}$ — yes
2. $\{(1,2), (2,3), (3,4)\}$ — no, missing $(2,1)$, etc
3. $\{(2,2), (3,3)\}$ — yes

Definitions (Antisymmetric, Asymmetric)

A relation R on A is **antisymmetric** iff whenever $(a, b) \in R$ and $(b, a) \in R$, $a = b$.

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Transitivity

Definition (Transitive)

A relation R on A is **transitive** iff $(a, c) \in R$ whenever $(a, b) \in R$ and $(b, c) \in R$.

$$\forall a, b, c \in A,\ (a, b) \in R \land (b, c) \in R \ \Rightarrow\ (a, c) \in R$$

Examples (Transitive)

Let $S = \{1, 2, 3, 4\}$. Which of the following relations is transitive?

1. $\{(1,1),(1,2),(2,2),(2,1),(3,3)\}$
2. $\{(1,3),(3,2),(2,1)\}$
3. $\{(2,4),(4,3),(2,3),(4,1)\}$

Transitivity means

> ***if*** *you can get from one point to another* ***in two hops,***
> ***then*** *you can also get there directly* ***in one hop***

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Definition (Transitive)

A relation R on A is **transitive** iff $(a,c) \in R$ whenever $(a,b) \in R$ and $(b,c) \in R$.

$$\forall a,b,c \in A,\ (a,b) \in R \land (b,c) \in R \Rightarrow (a,c) \in R$$

Examples (Transitive)

Let $S = \{1,2,3,4\}$. Which of the following relations is transitive?

1. $\{(1,1),(1,2),(2,2),(2,1),(3,3)\}$ — yes
2. $\{(1,3),(3,2),(2,1)\}$ — no, missing $(1,2)$, etc
3. $\{(2,4),(4,3),(2,3),(4,1)\}$ — no, missing $(2,1)$

Transitivity means

> ***if*** *you can get from one point to another* ***in two hops,***
> ***then*** *you can also get there directly* ***in one hop***

Example: Properties of Relations

Recall H and its relations:

$$
\begin{aligned}
H &= \{1, 2, 3, 4, 5\} \\
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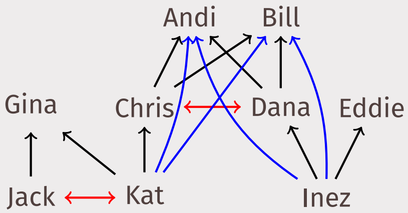
LE &= \{(1,1), (2,2), (3,3), (4,4), (5,5)\} \cup LT
\end{aligned}
$$

Of the relations S, LT, and LE, which are

- reflexive?
- symmetric?
- transitive?

Example: Properties of Relations

Recall the family tree example:



Of $ChildOf$ ($\uparrow$), $SiblingOf$ ($\uparrow$), and $DescendantOf$ ($\uparrow \cup \uparrow$), which are

- reflexive?
- symmetric?
- transitive?

Operations on Relations

Set Operations on Relations

Relations are sets, so we can apply set operations to them.

Suppose A and B are sets, and $R \subseteq A \times B$ and $S \subseteq A \times B$.

- $R \cup S$ relates a to b if either R or S relates them.
- $R \cap S$ relates a to b if both R and S relate them.
- $R - S$ relates a to b if R relates them and S does not.

That is:

$$R \cup S = \{(a, b) \mid (a, b) \in R \lor (a, b) \in S\}$$
$$R \cap S = \{(a, b) \mid (a, b) \in R \land (a, b) \in S\}$$
$$R - S = \{(a, b) \mid (a, b) \in R \land (a, b) \notin S\}$$

(It is uncommon to take a Cartesian product ($\times$) or power set ($\mathscr{P}$) of a relation.)

Example: Set Operations on the Red Line

$RL \subseteq St \times St$ is “one step on the Red Line”.

Let $St' = \{$South Station, Broadway, Andrew$\}$.

$RL \cap (St' \times St')$ represents the Red Line restricted to those stations.

Suppose $GL \subseteq St \times St$ represents the “one step” relation for the Green Line.

$RL \cup GL$ represents “reachable in one step using either the Red or Green Line”.

Suppose $BL \subseteq St \times St$ and $OL \subseteq St \times St$ represent the “one step” relations for the Blue and Orange Lines, respectively.

Let $T \subseteq St \times St$ be $T = RL \cup GL \cup BL \cup OL$.
T means “reachable in one step using the subway”.

Composing Relations

Definition (Composite)

Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The **composite** of R and S, written $S \circ R$, is defined as $\{(a, c) \mid (a, b) \in R,\ (b, c) \in S\}$. That is,

$$\forall a \in A,\ \forall c \in C,\ [(a, c) \in S \circ R \iff \exists b \in B,\ (a, b) \in R \wedge (b, c) \in S]$$

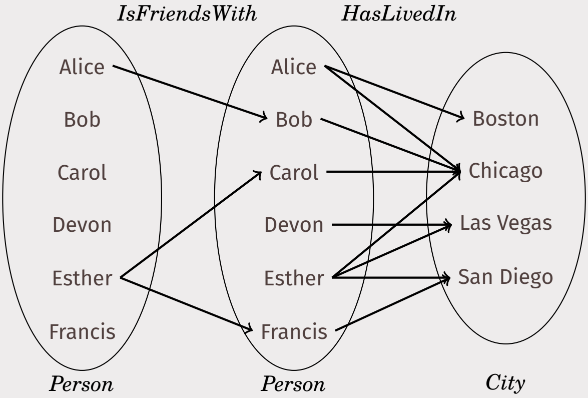
If you can get from a to b by R,
and you can get from b to c by S,
then you can get from a to c by $S \circ R$.

$S \circ R$ means "R first, then S"

Be careful of the order. The notation follows function composition notation, where $(g \circ f)(x) = g(f(x))$.

Example: Composing Relations

$$HasLivedIn \circ IsFriendsWith$$

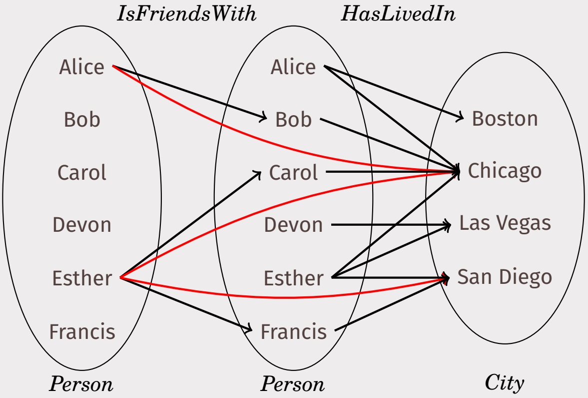


$$= \{(\text{Alice}, \text{Chicago}), (\text{Esther}, \text{Chicago}), (\text{Esther}, \text{San Diego})\}$$

Note that Bob, Carol, and Francis are not part of the result!

Example: Composing Relations

$$HasLivedIn \circ IsFriendsWith$$



$= \{(\text{Alice}, \text{Chicago}), (\text{Esther}, \text{Chicago}), (\text{Esther}, \text{San Diego})\}$

Note that Bob, Carol, and Francis are not part of the result!

Powers of Relations

Definition (Powers of a Relation)

Let A be a set, and $R \subseteq A \times A$. The **powers** of R are defined as follows:

- $R^1 = R$
- $R^{n+1} = R^n \circ R$

That is, $R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}$.

Example: Composition & Powers, Numbers

Recall $H = \{1, 2, 3, 4, 5\}$. We have

$$\begin{aligned} S &= \{(1,2), (2,3), (3,4), (4,5)\} \\ S^2 = S \circ S &= \{(1,3), (2,4), (3,5)\} \\ S^3 = S^2 \circ S &= \{(1,4), (2,5)\} \\ S^4 = S^3 \circ S &= \{(1,5)\} \\ S^5 = S^4 \circ S &= \emptyset \\ LT &= S \cup S^2 \cup S^3 \cup S^4 \end{aligned}$$

What properties does LT have, again?

Example: Composition & Powers, the Red Line

RL is “reachable in one step using the Red Line”, etc.

Let $T \subseteq S = RL \cup GL \cup BL \cup OL$.
T means “reachable in one step using the subway”.

$GL \circ RL$ means “reachable by riding one step on the Red Line, then one step on the Green Line”.

$RL^2 = RL \circ RL$ means “reachable by riding exactly two stops on the Red Line”.

$T^2 = T \circ T$ means “reachable by riding exactly two stops on the subway”.

$T \cup T^2$ means “reachable by riding one or two stops on the subway”.

Composition and Transitivity

Theorem

Let R be a relation on A. R is transitive iff $R^n \subseteq R$ for all $n \in \mathbb{Z}^+$.

Actually, it's even simpler: R is transitive iff $R^2 \subseteq R$.

Recall: R is transitive iff whenever R contains (a, b) and (b, c) it also contains (a, c). Those (a, c) pairs are exactly what $R^2 = R \circ R$ computes.

Observation

Let R be a relation on A. Even if R is not already transitive, we can create a transitive relation by taking

$$R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \cdots = \bigcup_{n=1}^{\infty} R^n$$

R^+ is called the transitive closure of R.

Closures of Relations

Definitions

Let R be a relation on A. That is, $R \subseteq A \times A$. Then

- the **reflexive closure** of R (wrt A) is the smallest relation R' such that $R \subseteq R'$ and R' is reflexive (wrt A)
- the **symmetric closure** of R is the smallest relation R' such that $R \subseteq R'$ and R' is symmetric
- the **transitive closure** of R is the smallest relation R' such that $R \subseteq R'$ and R' is transitive
- the **reflexive-reflexive closure** of R is the smallest relation R' such that $R \subseteq R'$ and R' is reflexive (wrt A) and transitive
- ...

Examples: Closures

Relations on $\mathbb{R}$:

- the reflexive closure of $<$ (wrt $\mathbb{R}$) is $\leq$
- the symmetric closure of $<$ is $\neq$
- the transitive closure of $<$ is $<$, because $<$ is already transitive
- the transitive closure of $\leq$ is $\leq$, because $\leq$ is already transitive

Recall: $H = \{1, 2, 3, 4, 5\}$, and $S = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$.

- the transitive closure of S is LT
- the reflexive closure of LT (wrt H) is LE
- the reflexive-transitive closure of S is LE

Examples: Closures (on the Red Line)

Recall $RL \subseteq St \times St$, meaning "reachable in one step on the Red Line".

- the reflexive closure of RL is
  $RL \cup \{(s,s) \mid s \in St\}$, meaning "reachable in zero or one steps"
- the symmetric closure of RL is
  RL, because RL is already symmetric
- the transitive closure of RL is
  $RL^+ = \bigcup_{k=1}^{\infty} RL^k$, meaning "reachable in one or more steps"
- the reflexive-transitive closure of RL is
  $RL^* = RL^+ \cup \{(s,s) \mid s \in RL\}$, "reachable in zero or more steps"
  In fact, $RL^* = RL^+$, because RL^+ is already reflexive. (Why?)

The transitive closure of "can get there in one step"
is "can get there in **one or more** steps".

The reflexive-transitive closure of "can get there in one step"
is "can get there in **zero or more** steps".

Calculating Closures

Let R be a relation on A.

- The reflexive closure of R is

$$R \cup \{(a,a) \mid a \in A\}$$

- The symmetric closure of R is

$$R \cup \{(b,a) \mid (a,b) \in R\}$$

- The transitive closure of R is

$$R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{n=1}^{\infty} R^n$$

Transitive Closure of a Directed Graph

If you think of a relation R as a directed graph, it is transitive if

- for every path from a to b containing two or more edges, there is an edge directly connecting a and b

To compute the transitive closure of R:

1. start with the graph of R
2. while there is a path from a to b with two edges
   but no edge from a to b
   add an edge from a to b

More Kinds of Relations

Partial Orders

Partial orders are a relaxed generalization of $\leq$-like relations.

Definition (Partial Order, Partial Ordering)

Let $\sqsubseteq$ be a relation on A. $\sqsubseteq$ is a **partial order** (aka, **partial ordering**) if it is

- reflexive, $\forall a \in A,\ a \sqsubseteq a$
- transitive, $\forall a, b, c \in A,\ (a \sqsubseteq b \wedge b \sqsubseteq c) \Rightarrow a \sqsubseteq c$
- and antisymmetric $\forall a, b \in A,\ (a \sqsubseteq b \wedge b \sqsubseteq a) \Rightarrow a = b$

Recall: R is antisymmetric if it doesn't contain any "mirrored pairs".
That is, if $a \neq b$, then R cannot contain both (a, b) and (b, a).

Definition (Partially Ordered Set, poset)

A set A with a partial order $\sqsubseteq$ on A is called a **partially ordered set**, often shortened to **poset**.

Examples: Partial Orders

A partial order is reflexive, transitive, and antisymmetric.

Examples

Which of the following are partial orders on H?

- $LT = \{(a,b) \mid a \in H,\ b \in H,\ a < b\}$
- $LE = \{(a,b) \mid a \in H,\ b \in H,\ a \leq b\}$
- $S = \{(1,2), (2,3), (3,4), (4,5)\}$
- $\{(1,1), (1,2), (2,2), (3,3), (4,4), (5,4), (5,5)\}$
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Examples: Partial Orders

A partial order is reflexive, transitive, and antisymmetric.

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- $LT = \{(a, b) \mid a \in H,\ b \in H,\ a < b\}$ — no, not reflexive
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- $\{(1,1), (1,2), (2,2), (3,3), (4,4), (5,4), (5,5)\}$ — yes!
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- $S = \{(1,2),(2,3),(3,4),(4,5)\}$ — no, not reflexive, not transitive
- $\{(1,1),(1,2),(2,2),(3,3),(4,4),(5,4),(5,5)\}$ — yes!
- $\{(1,1),(2,2),(3,3),(4,4),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ — yes!

Observation

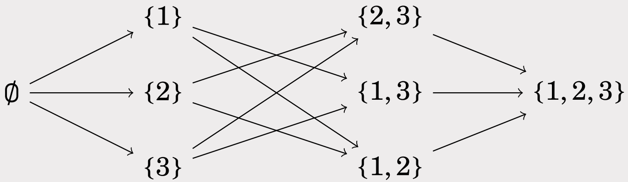
If $(S, \sqsubseteq)$ is a finite partially ordered set, then the elements of S can be written down so that all $\sqsubseteq$ arrows (except self-loops) go from left to right.

(This is related to a concept called topological sort.)

Examples: Partial Orders, the Inclusion Order

For any set A, $\subseteq$ is a partial order on $\mathscr{P}(A)$.

For example, here's the diagram* of $\subseteq$ on $\mathscr{P}(\{1, 2, 3\})$:



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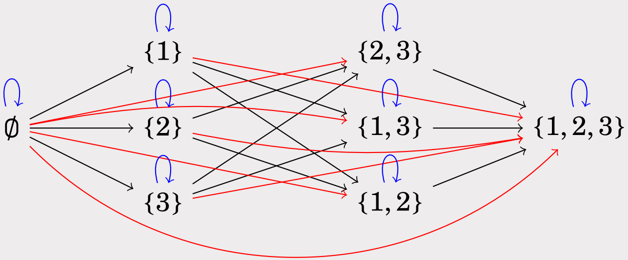

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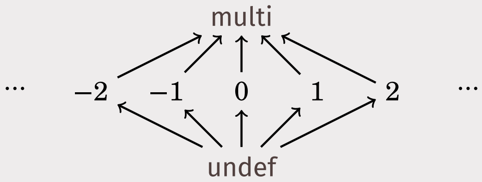
Examples: Partial Orders, the Flat Lattice

Here is a partial order (in fact, a lattice) on

$$\{\text{undef}, \text{multi}\} \cup \mathbb{Z}$$

This lattice is useful for **constant propagation** in an optimizing compiler.



Usually, undef is written $\bot$, and multi is written $\top$.

Comparable

Definition (Comparable)

Let $(S, \sqsubseteq)$ be a partially ordered set, and let $a \in S$ and $b \in S$.
Then a and b are **comparable** if either $a \sqsubseteq b$ or $b \sqsubseteq a$.
Otherwise they are **incomparable**.

Examples (H, R)

Are the following comparable in

$$R = \{(1,1),(1,2),(2,2),(3,3),(4,4),(5,4),(5,5)\}$$

- 1 and 2
- 3 and 4
- 4 and 5

Examples $(\mathscr{P}(\{1,2,3\}), \subseteq)$

Are the following comparable?

- $\emptyset$ and $\{1,2,3\}$
- $\{1,2\}$ and $\{1,3\}$
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- 3 and 4 — no
- 4 and 5 — yes

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- $\{1\}$ and $\{1,3\}$ — yes
- $\{2\}$ and $\{3\}$ — no

Total Orders

Total orders are a less relaxed generalization of $\leq$-like relations.

Definition (Total Order)

Let $\sqsubseteq$ be a relation on S. $\sqsubseteq$ is a **total order** (aka **linear order**) if

- $\sqsubseteq$ is a partial order, and
- every a and b in S are comparable (by $\sqsubseteq$)

Then $(S, \sqsubseteq)$ is a **totally ordered set**.

Examples

- $\leq$ and $\geq$ are total orders on $\mathbb{R}$ (and on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, etc)
- $\subseteq$ on $\mathscr{P}(S)$ is a partial order but generally not a total order
  (When is it a total order?)

Equivalence Relations

Equivalence relations are a generalization of $=$-like relations.

Definition (Equivalence Relation)

Let $\sim$ be a relation on A. It is an **equivalence relation** iff it is

- reflexive $\forall a \in A,\ a \sim a$
- symmetric $\forall a, b \in A,\ a \sim b \Rightarrow b \sim a$
- transitive $\forall a, b, c \in A,\ (a \sim b \land b \sim c) \Rightarrow a \sim c$

Then if $a \sim b$, we say a and b are **equivalent** (according to).

Examples

Some equivalence relations on the natural numbers ($\mathbb{N}$):

- $=$
- "have the same quotient when divided by 7"
- "have the same last digit" = "have same remainder if divided by 10"

Examples: Equivalence Relations on Strings

Some possible equivalence relations ($\equiv$) on **strings**:

- if they have both the same sequence of characters

  `"Apple"` $\not\equiv$ `"apple"`

- if, after both strings are case-folded (roughly, letters converted to lowercase), both have the same sequence of characters

  `"Apple"` $\equiv$ `"apple"`, `"Straße"` $\equiv$ `"strasse"`, `"two fish"` $\not\equiv$ `"twofish"`

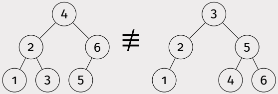
- if, after case folding and whitespace removal, both have the same sequence of characters

  `"Apple"` $\equiv$ `"apple"`, `"Straße"` $\equiv$ `"strasse"`, `"two fish"` $\equiv$ `"twofish"`

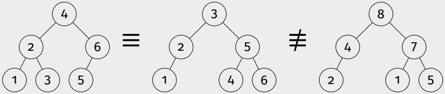
Examples: Equivalence Relations on Binary Trees

Some possible equivalence relations (≡) on **binary trees**:

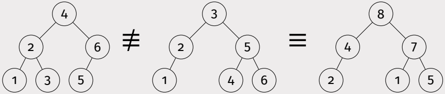
- if both have the same elements and the same tree structure



- if both have the same elements in the same left-to-right order (but maybe different structures)



- if both have the same tree structure (but maybe different elements)



Equivalence Classes

Definition (Equivalence Class)

Let $\sim$ be an equivalence relation on S, and let $a \in S$.
The **equivalence class** of a (under $\sim$), written $[a]_\sim$, is the set of all elements equivalent to a:

$$[a]_\sim = \{b \mid a \sim b\}$$

(If the relation is clear from context, we can omit the subscript and just write $[a]$.)

Examples

- For $\mathbb{N}$ with the "have the same last digit" relation:
  $[24] =$
- For $\mathbb{N}$ with the "have the same quotient when divided by 7" relation:
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- For $\mathbb{N}$ with the "have the same quotient when divided by 7" relation:
  $[24] = \{21, 22, 23, \mathbf{24}, 25, 26, 27\}$ (quotient is 3)

Equivalence Classes

Facts about Equivalence Classes

Let $\sim$ be a equivalence relation on S. Then

- $a \in [a]$ for every $a \in S$
- $a \sim b \iff [a] = [b]$
- $a \nsim b \iff [a] \cap [b] = \emptyset$
- $\{[a] \mid a \in S\}$ is a partition of S

Example

The equivalence relation "ends with the same digit" partitions $\mathbb{N}$ into 10 equivalence classes:

$$\{\{0, 10, 20, \dots\}, \{1, 11, 21, \dots\}, \{2, 12, 32, \dots\}, \dots, \{9, 19, 29, \dots\}\}$$

Topic List

- relations as sets of pairs
- digraph (directed graph) representation
- relation properties: reflexive, symmetric, antisymmetric, transitive
- relation composition ($\circ$), powers
- {reflexive,symmetric,transitive}-closure
- partial order, total order
- equivalence relation, equivalent
- equivalence classes