Functions

CS 220 — Applied Discrete Mathematics

{February 26, March 3}, 2025

Functions

Definition (Function)

Let A and B be sets. A **function** (or **total function**) from A to B is a relation where each element of A is related to exactly one element of B.

We write $f : A \to B$ to indicate that f is a function from A to B.

If $(a, b) \in f$, then we say $f(a) = b$.

Definitions (Domain, Codomain)

Let $f : A \to B$.

- A is called the **domain** of f. It is the set of arguments (inputs).
- B is called the **codomain** of f. It is the set of results (outputs).

We say f **maps** A to B.

Range

Definition (Range)

Let $f : A \to B$. The **range** of f is the set $\{f(a) \mid a \in A\}$.

That is, the range of f is all the outputs that f actually produces.

Examples

What is the range of each of the following $\mathbb{R} \to \mathbb{R}$ functions?

- $f_1(x) = 2x - 3$
- $f_2(x) = x^2 + 1$
- $f_3(x) = \sin(x)$
- $f_4(x) = e^x$

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- $f_1(x) = 2x - 3$ $\quad \mathbb{R}$
- $f_2(x) = x^2 + 1$ $\quad [1, \infty)$
- $f_3(x) = \sin(x)$ $\quad [-1, 1]$
- $f_4(x) = e^x$ $\quad (0, \infty)$

Equality of Functions

Equality

Let $f, g : A \to B$. Then f is equal to g (as functions) if $f = g$ (as sets).
That means that $f(a) = g(a)$ for all $a \in A$.

Examples

$$f_1(x) = 2x \qquad g_1(x) = x^2 + 6x + 9$$

$$f_2(x) = x + x \qquad g_2(x) = (x+3)^2$$

This notion of equality is called **extensional equality**.
(In contrast, **intensional equality** means "defined the same way", roughly.)

Properties of Functions

Injective ("one-to-one") Functions

Definition (Injective)

Let $f : A \to B$. Then f is **injective** (or "one-to-one") if f maps different arguments to different results. That is:

$$f \text{ is injective} \iff \forall a, a' \in A,\ a \neq a' \Rightarrow f(a) \neq f(a')$$

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Examples

Consider the following $\mathbb{R} \to \mathbb{R}$ functions:

- $f_1(x) = 2x - 3$ — injective
- $f_2(x) = x^2 + 1$ — not injective, $f(1) = f(-1) = 2$
- $f_3(x) = \sin(x)$ — not injective, $f(0) = f(2\pi) = 0$
- $f_4(x) = e^x$ — injective

Surjective ("onto") Functions

Definition (Surjective)

Let $f : A \to B$. Then f is **surjective** (or "onto") if the set of f's results covers all of B. That is:

$$f \text{ is surjective} \iff \forall b \in B,\ \exists a \in A,\ f(a) = b$$

A function is surjective if its range is equal to its codomain.

Examples

Consider the following $\mathbb{R} \to \mathbb{R}$ functions:

- $f_1(x) = 2x - 3$
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Examples

Consider the following $\mathbb{R} \to \mathbb{R}$ functions:

- $f_1(x) = 2x - 3$ — surjective, range is $\mathbb{R}$
- $f_2(x) = x^2 + 1$ — not surjective, range is $[-1, \infty)$
- $f_3(x) = \sin(x)$ — not surjective, range is $[-1, 1]$
- $f_4(x) = e^x$ — not surjective, range is $(0, \infty)$

Bijective Functions

Definition (Bijective)

Let $f : A \to B$. Then f is **bijective** iff it is both injective and surjective.

That is, each element in A is matched with exactly one element in B, and vice versa.

A **bijection** is also called a **one-to-one correspondence** (!).

Examples

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Examples

Consider the following $\mathbb{R} \to \mathbb{R}$ functions:

- $f_1(x) = 2x - 3$ — bijective
- $f_2(x) = x^2 + 1$ — not bijective (neither injective nor surjective)
- $f_3(x) = \sin(x)$ — not bijective (neither injective nor surjective)
- $f_4(x) = e^x$ — not bijective (not surjective)

Exercise: Classify Functions

Which of the following $\mathbb{Z} \to \mathbb{Z}$ functions are injective? surjective? bijective?

- $f(n) = 2n + 1$
- $g(n) = n^2 - 1$
- $h(n) = 5 - n$

Which of the following $H \to H$ functions are injective? surjective? bijective? (Recall $H = \{1, 2, 3, 4, 5\}$.)

- $p = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$
- $q = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$
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Functions and the Cardinality of Sets

Let $H = \{1, 2, 3, 4, 5\}$ and let $B = \{0, 1\}$.

- Is there a bijective function $H \to B$?

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  No. We can't match them up one-to-one.

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- Is there an injective function $H \to B$?

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  No, we "run out" of B first.

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- Is there an injective function $H \to B$?

  No, we "run out" of B first.
- Is there a surjective function $H \to B$?

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- Is there a surjective function $H \to B$?
  Yes, there are several.

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- Is there an injective function $H \to B$?
  No, we "run out" of B first.
- Is there a surjective function $H \to B$?
  Yes, there are several.

Observation

Let A, B be finite sets. Then

- There is a bijective function $A \to B$ iff $|A| = |B|$.
- There is an injective function $A \to B$ iff $|A| \leq |B|$.
- There is a surjective function $A \to B$ iff $|A| \geq |B|$.

Functions and the Cardinality of Sets

Definition (Cardinality)

$$|A| \leq |B| \iff \text{there is some injective function } A \to B$$
$$|A| = |B| \iff \text{there is some bijective function } A \to B$$
$$|A| > |B| \iff \text{there is no injective function } A \to B$$

Definition (Countable)

A set is **countable** if it has the same cardinality as $\mathbb{N}$.
That is, a set is countable if there is a bijection from it to $\mathbb{N}$.

Definition (Uncountable)

A set is **uncountable** if its cardinality is larger than that of $\mathbb{N}$.
That is, a set if uncountable if there is no injection from it to $\mathbb{N}$.

Functions on Totally Ordered Sets

Definitions (Strictly Increasing, Strictly Decreasing)

Let $f : \mathbb{R} \to \mathbb{R}$.

- f is called **strictly increasing** if

$$\forall x, x' \in \mathbb{R},\ x < x' \Rightarrow f(x) < f(x')$$

- f is called **strictly decreasing** if

$$\forall x, x' \in \mathbb{R},\ x < x' \Rightarrow f(x) > f(x')$$

All strictly increasing and strictly decreasing functions are injective.

The definitions generalize to any functions from a totally ordered set to another totally ordered set.

Functions on Partially Ordered Sets

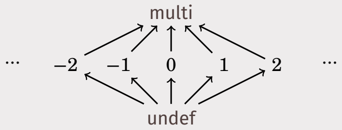
Definition (Monotone, Monotonic)

Let $f : A \to B$. Let $\sqsubseteq_A$ be a partial order on A and $\sqsubseteq_B$ be a partial order on B. Then f is called **monotone** or **monotonic** or **monotone non-decreasing** if

$$\forall a, a' \in A,\ a \sqsubseteq_A a' \Rightarrow f(a) \sqsubseteq_B f(a')$$

Example (Application)

Optimizing compilers use the **monotone data flow analysis framework**, which expresses **analyses** using monotone functions. More precise inputs lead to more precise outputs, and each analysis step can make progress, but it cannot change its mind.



Partial Functions

Definition (Partial Function, Total Function)

- A **partial function** is a relation from A to B where each element of A is related to *at most one* element of B. That is, there *may* be elements of the domain for which the partial function is **undefined**.
- A **total function** is a relation from A to B where each element of A is related to *exactly one* element of B. That is, a total function is defined everywhere on its domain. Every total function is also a partial function.

Examples

- $f(x) = \frac{1}{x+1}$ is a partial function on $\mathbb{R} \to \mathbb{R}$; it is not total
- $h = \{(2n, n) \mid n \in \mathbb{N}\}$ is a partial function on $\mathbb{N} \to \mathbb{N}$; it is not total

In math, “function” may mean partial or total function, depending on context.
In this class, “function” means total function.

Operations on Functions

Function Composition

Definition (Composition)

Let $f : A \to B$ and $g : B \to C$. The (function) composition of f with g is written $g \circ f$, and it is defined as

$$(g \circ f)(a) = g(f(a))$$

(Function composition is just relation composition on relations that are functions.)

Example

Suppose we have $f, g : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = 2x - 1 \qquad\qquad g(x) = x^2 + 4$$

Then we can calculate $(g \circ f)(x)$ as follows:

$$(g \circ f)(x) = g(f(x)) = g(2x - 1) = (2x - 1)^2 + 4 = 4x^2 - 4x + 5$$

Image and Pre-Image

Definition (Image)

Let $f : A \to B$, and let $X \subseteq A$.

The **image** of X under f, written $f(X)$, is

$$f(X) = \{f(x) \mid x \in X\}$$

That is, the image of X is the set of all *outputs* produced by some *input* from X.

Definition (Pre-Image)

Let $f : A \to B$, and let $Y \subseteq B$.

The **pre-image** of Y under f, written $f^{-1}(Y)$, is

$$f^{-1}(Y) = \{x \mid x \in A,\ f(x) \in Y\}$$

That is, the pre-image of Y is the set of all *inputs* that produce some *output* in Y.

Image and Pre-Image

Recall: If $f : A \to B$, and $X \subseteq A$ and $Y \subseteq B$, then

$f(X) = \{f(x) \mid x \in X\}$	all outputs from inputs from X
$f^{-1}(Y) = \{x \mid x \in A,\ f(x) \in Y\}$	all inputs that produce outputs in Y

Examples

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$.

- $f(\{1,2,3\}) =$
- $f(\{-2,2\}) =$
- $f([2,5]) =$
- $f^{-1}(\{1\}) =$
- $f^{-1}(\{-2\}) =$
- $f^{-1}(\{-4,9\}) =$

Image and Pre-Image

Recall: If $f : A \to B$, and $X \subseteq A$ and $Y \subseteq B$, then

$f(X) = \{f(x) \mid x \in X\}$	all outputs from inputs from X
$f^{-1}(Y) = \{x \mid x \in A,\ f(x) \in Y\}$	all inputs that produce outputs in Y

Examples

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$.

- $f(\{1, 2, 3\}) = \{1, 4, 9\}$
- $f(\{-2, 2\}) = \{4\}$
- $f([2, 5]) = [4, 25]$
- $f^{-1}(\{1\}) =$
- $f^{-1}(\{-2\}) =$
- $f^{-1}(\{-4, 9\}) =$

Image and Pre-Image

Recall: If $f : A \to B$, and $X \subseteq A$ and $Y \subseteq B$, then

$f(X) = \{f(x) \mid x \in X\}$	all outputs from inputs from X
$f^{-1}(Y) = \{x \mid x \in A,\ f(x) \in Y\}$	all inputs that produce outputs in Y

Examples

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$.

- $f(\{1,2,3\}) = \{1,4,9\}$
- $f(\{-2,2\}) = \{4\}$
- $f([2,5]) = [4,25]$
- $f^{-1}(\{1\}) = \{-1,1\}$
- $f^{-1}(\{-2\}) = \emptyset$ okay!
- $f^{-1}(\{-4,9\}) = \{-3,3\}$ okay!

Note: the image and pre-image operations take a set and produce a set.
Both are defined for all functions, but the pre-image of some sets might be empty.

Inverse Function

Definition (Inverse)

Let $f : A \to B$, and suppose f is a bijection. Then it has an inverse function, written $f^{-1} : B \to A$, defined by

$$f^{-1}(y) = x \quad \text{such that } f(x) = y$$

(The notation is the same as the pre-image.)

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x + 1$.

This f is bijective, and we can write its inverse as $f^{-1}(y) = \frac{y-1}{2}$.

For example, $f(5) = 11$, and $f^{-1}(11) = 5$.

Real-Valued Functions

Definition (Real-Valued Function)

A function f is called a **real-valued function** if its codomain is $\mathbb{R}$.
That is, $f : A \to \mathbb{R}$, for some A.

Definition (Sum and Product of Functions)

Let $f, g : A \to \mathbb{R}$. Then $f + g : A \to \mathbb{R}$ and $f \cdot g : A \to \mathbb{R}$:

$$(f + g)(x) = f(x) + g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Important Functions

The Identity Function

Definition (Identity Function)

The **identity function** on a set A, written $\mathrm{id}_A : A \to A$, is defined by

$$\mathrm{id}_A(a) = a$$

That is, its result is always the same as its argument.

Examples

- $\mathrm{id}_{\mathbb{R}}(2) = 2$
- $\mathrm{id}_{\mathbb{R}}(1.7) = 1.7$
- $\mathrm{id}_{\mathbb{R}}(\pi) = \pi$

id_A is reflexive when viewed as a relation on A.

Floor and Ceiling

Definitions (Floor, Ceiling)

- The **floor** function is a function $\mathbb{R} \to \mathbb{Z}$, written $\lfloor x \rfloor$.
  $\lfloor x \rfloor$ is the greatest integer less than or equal to x.
  (This is sometimes called "rounding towards $-\infty$".)
- The **ceiling** function is a function $\mathbb{R} \to \mathbb{Z}$, written $\lceil x \rceil$.
  $\lceil x \rceil$ is the least integer greater than or equal to x.
  (This is sometimes called "rounding towards $+\infty$".)

Examples

$$\lfloor 6 \rfloor = 6 \qquad \lceil 6 \rceil = 6$$
$$\lfloor 1.2 \rfloor = 1 \qquad \lceil 1.2 \rceil = 2$$
$$\lfloor -7.4 \rfloor = -8 \qquad \lceil -7.4 \rceil = -7$$

Min and Max

There are partial functions $\min : \mathscr{P}(\mathbb{R}) \to \mathbb{R}$ and $\max : \mathscr{P}(\mathbb{R}) \to \mathbb{R}$ that take a set of numbers and return the minimum or maximum element, respectively (if it exists).

Examples

- $\min\{1, 2, 3\} = 1$
- $\max\{1, 2, 3\} = 3$
- $\min\left\{(n+3)^2 \,\middle|\, n \in \mathbb{N}\right\} = 9$
- $\max\left\{x \,\middle|\, x \in \mathbb{R},\ x^2 \leq 2\right\} = \sqrt{2}$

The following are undefined:

- $\min(\emptyset),\ \max(\emptyset)$
- $\min(\mathbb{R}),\ \max(\mathbb{R})$
- $\max\left\{x \,\middle|\, x \in \mathbb{R},\ x^2 < 2\right\}$

$\min$ and $\max$ are also sometimes written as indexed computations:

$$\min_{n \in \mathbb{N}} (n-2)^2 = \min\left\{(n-2)^2 \,\middle|\, n \in \mathbb{N}\right\}$$

Using Min and Max

Mathematicians often define things as “the greatest *blah* such that *blah*” (or “least”). We can write those definitions in terms of $\min$, $\max$, and sets.

Example

Let’s re-examine the floor and ceiling functions:

$$\begin{aligned}\lfloor x \rfloor &= \text{“the greatest integer less than or equal to } x\text{”} \\ &= \max\,\{z \mid z \in \mathbb{Z},\ z \le x\}\end{aligned}$$

$$\begin{aligned}\lceil x \rceil &= \text{“the least integer greater than or equal to } x\text{”} \\ &= \min\,\{z \mid z \in \mathbb{Z},\ z \ge x\}\end{aligned}$$

Min and Max for Partial Orders

We can generalize $\min$ and $\max$ to any partially-ordered set.

Definition (Minimum, Maximum)

Let $(A, \sqsubseteq)$ be a partially-ordered set, and let $X \subseteq A$. The **minimum** and **maximum** of X are defined as

$$\min_{\sqsubseteq}(X) = x \iff x \in X \land (\forall a \in X,\ x \sqsubseteq a)$$
$$\max_{\sqsubseteq}(X) = x \iff x \in X \land (\forall a \in X,\ a \sqsubseteq x)$$

Such an x might not exist, but if it does then it is unique. (Why?)

Example

$$\begin{aligned}\text{reflexive-closure}_A(R) &= \text{“the smallest relation containing } R \text{ that is reflexive on } A\text{”} \\ &= \min_{\subseteq}\{R' \mid R' \subseteq A \times A,\ R \subseteq R',\ \mathrm{id}_A \subseteq R'\}\end{aligned}$$

Argmin and Argmax

Definitions (Argmin, Argmax)

Let $f : A \to \mathbb{R}$. Then

- $\arg\min f$ returns the *input value* that minimizes the function's *output*
- $\arg\max f$ returns the *input value* that maximizes the function's *output*

The result might not be defined, if the function does not have a minimum/maximum output, or if the minimum/maximum output is produced by multiple inputs.

Example

$\arg\min$ and $\arg\max$ can also be written as indexed computations:

$$\arg\max_{x \in [0, 2\pi]} (\sin x) = \frac{\pi}{2}$$

because $\sin x$ achieves its maximum value 1 at $x = \frac{\pi}{2}$ (within the domain $[0, 2\pi]$).

Defining Functions

Defining Functions by Cases

A function can be defined by multiple cases, each of which specifies a condition and a result. (This is the math version of `if-then-else`.)

The cases should not overlap—or if they do, they should produce consistent results on the overlapping points.

Example

The following notation is usually used for functions defined by cases:

$$f(x) = \begin{cases} x & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \end{cases}$$

To evaluate f:

- $f(1)$ falls into the first case, so $f(1) = 1$
- $f(-2)$ falls into the second case, so $f(-2) = 0$

Defining Functions by Patterns

Another way of defining a function is to give one or more (non-overlapping) definitions using "pattern matching" arguments.

Example

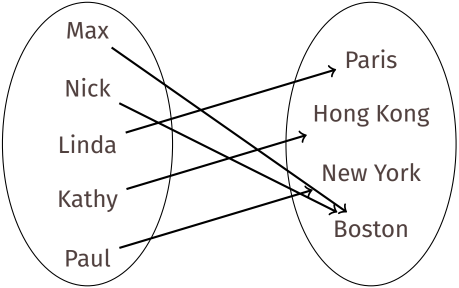
We could define a function $f : \mathbb{N} \to \mathbb{N}$ that acts differently depending on whether the argument is odd or even:

$$\begin{aligned} f(2k) &= k^3 \\ f(2k+1) &= 2^k \end{aligned}$$

To evaluate f:

- $f(7)$ can be expressed as $f(2 \cdot \mathbf{3} + 1)$, so the second case matches with $k = \mathbf{3}$, so $f(7) = 2^{\mathbf{3}} = 8$
- $f(8)$ can be expressed as $f(2 \cdot \mathbf{4})$, so the first case matches with $k = \mathbf{4}$, so $f(8) = \mathbf{4}^3 = 64$

Other Ways of Representing Functions



x	$\text{LivesIn}(x)$
Max	Boston
Nick	Boston
Linda	Paris
Kathy	Hong Kong
Paul	New York

Anonymous Functions

In mathematics, functions are usually defined by giving them a name. But sometimes it is useful to refer to a function without a name.

- "maps-to notation"
  For example, $(x \mapsto x^2)$ is the function that squares its argument.
- "λ (lambda) notation"
  For example, $(\lambda x.\ 2x + 1)$ is the function that takes a number, doubles it, and adds 1.

Homomorphisms

Homomorphisms

Definition (Homomorphism)

Let A and B be sets, and let $c_A : A \times A \rightarrow A$ and $c_B : B \times B \rightarrow B$ be binary functions on A and B, respectively, and let $f : A \rightarrow B$,.

The function f is a **homomorphism** (wrt c_A, c_B) if

$$\forall a_1, a_2 \in A,\ \ f(c_A(a_1, a_2)) = c_B(f(a_1), f(a_2))$$

(We can generalize from one binary operation c to multiple n-ary operations.)

In this situation:

- A and B are two *representations* of the same *information*.
- There is some some computation c to perform.
- The final result is represented as an element of B.
- We have the freedom of two options:
  - perform the computation in A using c_A, then translate to B using f
  - translate to B using f, then perform the computation in B using c_B

This freedom is the invisible **foundation of all computation**.

Isomorphisms

Definition (Isomorphism)

An **isomorphism** is a bijective homomorphism.

Definition (Isomorphic)

Two sets are **isomorphic** if there is some isomorphism between them.

In this situation, additionally:

- We have the freedom to translate back and forth between A and B.
  The translation does not lose information.

In practice, we often have a surjective homomorphism instead of an isomorphism, so there is no unique inverse, but we can pick an element for the backwards translation.

Examples: Homomorphisms and Isomorphism

Example (Logarithms)

$$\begin{array}{ccc} A = \mathbb{R}^+ & & B = \mathbb{R} \\ c_A(x,y) = x \cdot y & \xrightarrow{\log} & c_B(x',y') = x' + y' \\ d_A(x) = \sqrt[n]{x} & & d_B(x') = \frac{x'}{n} \end{array}$$

https://www.youtube.com/watch?v=habHK6wLkic

Example (3D Geometry)

$$\begin{array}{ccc} A = \text{affine transformations} & & B = 4 \times 4 \text{ matrices on } \mathbb{R} \\ c_A(X,Y) = \text{compose } X \text{ and } Y & \xrightarrow{\begin{bmatrix} x_x & x_y & x_z & x_0 \\ y_x & y_y & y_z & y_0 \\ z_x & z_y & z_z & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} & c_B(X',Y') = X'Y' \\ d_A(X) = \text{transform point } P \text{ by } X & & d_B(X') = X'P \end{array}$$

Hilbert's Hotel

Professor Hilbert's Hotel

Professor Hilbert runs a special hotel, with infinitely many rooms.

- That is, each room has a distinct room number ($\in \mathbb{N}$) and for every $n \in \mathbb{N}$ there is a room labeled n.
- The hotel also has an excellent PA system.

One day, the hotel happens to be completely full.
A new person enters the hotel and asks for a room.

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A new person enters the hotel and asks for a room.

Hilbert says "Certainly!" He announces over the PA system:

> *Pardon the disruption, every guest please change rooms.*
> *Move to the room number of your current room plus one.*

The guests move, and the new person gets the now-empty room Zero, so everyone has a room.

Professor Hilbert's Hotel

Then an infinitely long bus pulls up.
The driver comes to the reception desk.

> *My bus has a seat for every* $n \in \mathbb{N}$, *and each seat has a passenger. Can you find rooms for us all?*

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Hilbert says "Certainly!" He gets on the PA system again:

> *Apologies, please change rooms again.*
> *Move to the room number that is double your current room number.*

The guests move, and he says to the driver:

> *Each of your passengers can take the room whose number is one more than twice their seat number.*

And so there was room for everyone.

Professor Hilbert's Hotel

Then an infinite caravan of infinitely long buses pulls up...

$$\begin{array}{ccccc} 0,0 & 0,1 & 0,2 & 0,3 & \ldots \\ 1,0 & 1,1 & 1,2 & 1,3 & \ldots \\ 2,0 & 2,1 & 2,2 & 2,3 & \ldots \\ 3,0 & 3,1 & 3,2 & 3,3 & \ldots \\ \ldots & & & & \end{array}$$

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Hilbert says "Certainly!"
The formula is a bit complicated, but passengers get admitted in groups according to the sum of their bus number and their seat number:

$$\begin{array}{llll} 0,0 & & & \\ 0,1 & 1,0 & & \\ 0,2 & 1,1 & 2,0 & \\ 0,3 & 1,2 & 2,1 & 3,0 \ldots \end{array}$$

And so there is space for everyone.

Professor Cantor's Shift

Professor Hilbert's shift ends, and Professor Cantor takes over.

A spaceship lands nearby. The pilot says:

> *My ship has a passenger for every number in* $\mathbb{R}$.
>
> *Do you have room for everyone?*

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> *My ship has a passenger for every number in $\mathbb{R}$.*
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> *Do you have room for everyone?*

Cantor says "No."

Professor Cantor's Shift

The driver says

> *Okay, that figures. Most of them are unimportant, anyway.*
> *My important passengers are all in the unit interval* $[0, 1)$.
>
> *Do you at least have room for them?*

Professor Cantor's Shift

The driver says

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>
> *Do you at least have room for them?*

Cantor says "No."

Professor Cantor's Shift

The driver says

> *My most important passengers, the VIPs, are all in the unit interval* $[0, 1)$ *and their decimal expansions only include the digits* 0 *and* 1.
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> *Surely you have room for them?*

Professor Cantor's Shift

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Cantor says "No."

Suppose I *could* fit everyone.
The assignment would look like this:

$$0 \mapsto 0\,.\,\mathbf{0}\;0\;0\;0\;1\ldots$$
$$1 \mapsto 0\,.\,0\;\mathbf{1}\;1\;0\;1\ldots$$
$$2 \mapsto 0\,.\,1\;0\;\mathbf{1}\;0\;1\ldots$$
$$3 \mapsto 0\,.\,1\;0\;1\;\mathbf{0}\;0\ldots$$
$$4 \mapsto 0\,.\,0\;1\;1\;1\;\mathbf{0}\ldots$$

But consider the main diagonal.
What about the number formed by flipping each digit?

$$0.10011\ldots$$

That person can't be on the list.

That is, I can show that every possible "solution" is broken. It can't be done.

Infinite Cardinalities

$$|\mathbb{N}| = |\mathbb{N} + \{0\}|$$
$$|\mathbb{N}| = |\mathbb{N} + \mathbb{N}|$$
$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

$$|\mathbb{N}| < |\mathbb{R}|$$
$$|\mathbb{N}| < |[0, 1)|$$
$$|\mathbb{N}| < |\mathscr{P}(\mathbb{N})|$$

where $A + B = (\{0\} \times A) \cup (\{1\} \times B)$ — the "tagged union" of A and B

Topic List

- (total) function, domain, codomain, range
- injective, surjective, bijective
- partial vs total function
- strictly increasing/decreasing, ~~monotone~~
- function composition ($\circ$)
- image, pre-image
- inverse
- identity function
- floor ($\lfloor\ \rfloor$), ceiling ($\lceil\ \rceil$)
- min, max, ~~arg min, arg max~~
- function notations: cases, pattern-matching, tables, etc
- ~~homomorphisms, isomorphisms~~
- Hilbert's Hotel, cardinalities of infinite sets