Counting

CS 220 — Applied Discrete Mathematics

April {23, 28}, 2025

Counting Problems

Counting problems are of the following kind:

- "How many different 8-letter passwords are there?"
- "How many possible ways are there to pick 11 soccer players out of a 20-player team?"

Counting is the basis for computing probabilities of discrete events.

- "What is the probability of winning the lottery?"

Generative Processes and Outcomes

I'll use the term "**generative process**" (or just "process") to cover

- a **task** that can be accomplished in multiple ways,
- a **choice** between multiple options,
- etc.

An **outcome** is the end result of a generative process.

Examples

Generative Process: Pick an 8-letter password.
Outcomes: "aaaaaaaa", "coolmath", "mysecret", ...

Generative Process: Pick a set of two elements from $\{A, B, C\}$.
Outcomes: $\{A, B\}$, $\{A, C\}$, $\{B, C\}$

Decomposition of Generative Processes

Complex processes can often be decomposed into simpler processes.

Examples

Generative Process: Pick an 8-letter password.
Decomposition:

- Pick one letter to be the 1st letter of the password.
- Pick one letter to be the 2nd letter of the password.
- ...and so on, up to the 8th letter.

Generative Process: Pick a set of two elements from $\{A, B, C\}$.
Decomposition:

- Pick an ordered list (or tuple) of two distinct elements from $\{A, B, C\}$.
- Adjust for ordering (list vs set).

Basic Principles

The Addition Principle

The Addition Principle, aka the Sum Rule

Suppose that a process can be done in two ways—that is, exactly **one** way is chosen, either the first or the second, but not both.

If the first way has n_1 outcomes, and the second way has n_2 outcomes, and the outcomes do not overlap, then there are $n_1 + n_2$ total outcomes.

Examples

- Pick either a digit (0-9) or a letter (A-Z). How many outcomes are there?
- The dean will award a free phone to either a CS student or a math student. There are 530 CS students and 264 math students (but there are some double-majors). How many possible recipients are there?

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  Solution: There are $10 + 26 = 36$ total outcomes (choices).
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- The dean will award a free phone to either a CS student or a math student. There are 530 CS students and 264 math students (but there are some double-majors). How many possible recipients are there?

  Error: The addition principle **does not apply**, since there is overlap between the CS students and the math students.

The Multiplication Principle

The Multiplication Principle, aka the Product Rule

Suppose that a process can be broken down into two parts, where **both** parts must be performed and their results are **simply combined**.

If the first part has n_1 outcomes and then the second part has n_2 outcomes, then the combined process has $n_1 n_2$ total outcomes.

Examples

- How many codes could we form from exactly two English letters?
- If we roll a red die and a blue die (both dice have six sides), how many different numbers can they sum to?

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- How many codes could we form from exactly two English letters?

  Solution: There are 26 outcomes for the first letter, and 26 outcomes for the second letter. So there are $26 \cdot 26 = 676$ total outcomes.
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- If we roll a red die and a blue die (both dice have six sides), how many different numbers can they sum to?

  Error: The multiplication principle **does not apply** to this problem, since different pairs of dice results can lead to the same final outcome (sum). That is, the results are not "simply combined".

The Multiplication Principle

We can still use the multiplication principle if the second task depends on the outcome of the first, as long as the number of outcomes doesn't vary.

Example

How many codes can we form with two distinct letters?

Decomposition:

- Step 1: Pick the first letter.
  There are 26 choices.
- Step 2: Pick the second letter, excluding the letter picked in step 1.
  There are $26 - 1 = 25$ choices.
  (The set of available letters depends on step 1, but the number does not.)

So there are $26 \cdot 25 = 650$ total outcomes.

The Subtraction Principle

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Suppose that a process can be described as follows: first perform a sub-process with n_1 outcomes, but then **exclude** every outcome that can be produced by a second sub-process (with n_2 outcomes).

If every outcome of the second sub-process is also an outcome of the first, then there are $n_1 - n_2$ total outcomes.

Example

- How many codes can be formed from two digits (0-9), if the digits cannot both be even? (That is: 12, 63, and 55 are okay, but 24 is not.)

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- How many codes can be formed from two digits (0-9), if the digits cannot both be even? (That is: 12, 63, and 55 are okay, but 24 is not.)

  Solution: We use the subtraction principle:

  - By the product rule, there are 10^2 two-digit codes.
  - There are 5 even digits, so by the product rule there are 5^2 codes of two even digits. Every two-even-digit code is a two-digit code.
  - So there are $10^2 - 5^2 = 75$ two-digit codes not having two even digits.

The Quotient Principle

The Quotient Principle, aka the k-to-1 Rule

Suppose that every outcome of a process corresponds to exactly k different outcomes of a sub-process, and every sub-process outcome corresponds to exactly one main outcome.

If the sub-process has n outcomes, then the main process has $\frac{n}{k}$ outcomes.

Example

- How many two-digit codes are there where the digits are in *ascending order* (that is, the second digit is greater than the first)?

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Example

- How many two-digit codes are there where the digits are in *ascending order* (that is, the second digit is greater than the first)?

  Solution: We use the quotient principle with $k = 2$:
  - There are $10 \cdot 9 = 90$ two-digit codes where the digits are *different*.
  - Every *ascending-order code* corresponds to exactly 2 *different-digits code*. For example, 47 corresponds to 47 and 74.
  - So there are $\frac{10 \cdot 9}{2} = 45$ two-digit codes with the digits in ascending order.

Counting Principles and Sets

Addition Principle for Sets

Let A_1, A_2 be finite, disjoint sets.
Then the number of ways to choose one element from either set is

$$|A_1 \cup A_2| = |A_1| + |A_2|$$

Multiplication Principle for Sets

Let A, B_1, B_2 be finite sets, and let $f : B_1 \times B_2 \to A$ be bijective.
Then the number of ways to choose an element of A corresponds to the number of ways to choose an ordered pair from $B_1 \times B_2$:

$$|A| = |B_1 \times B_2| = |B_1| \cdot |B_2|$$

The bijection f represents the "simple combination of results".

Counting Principles and Sets

Subtraction Principle for Sets

Let A, X be finite sets where $X \subseteq A$.

Then the number of ways to choose an element from $A - X$ is

$$|A - X| = |A| - |X|$$

Quotient Principle for Sets

Let A, B be finite sets, let $k \in \mathbb{Z}^+$, and let $f : A \to B$ have the property that $\left| f^{-1}(\{b\}) \right| = k$ for every $b \in B$. Then the number of ways to select one element from B is

$$|B| = \frac{|A|}{k}$$

The function f is called a **k-to-1 correspondence**.

Exercise: Sums and Products

- How many different license plates are there that contain exactly three letters followed by two digits?
- How many different codes are there if a code is either three digits or two letters?
- How many codes are there if a code contains one or two letters followed by between two and four digits? For example: `D55`, `KF3930`, `AA123`, and `CS220`.

Inclusion-Exclusion

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- **Process 1:** Construct a bit string of length 8 that starts with a 1. There is 1 choice for the first bit and 2 choices for each of bits 2–8. So there are $1 \cdot 2^7 = 128$ outcomes.
- **Process 2:** Construct a bit string of length 8 the ends with 00. There are 2 choices for each of bits 1–6 and 1 choice for bits 7–8. So there are $2^6 \cdot 1^2 = 64$ outcomes.

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~~So by the sum rule, there are 192 total possibilities.~~ This is **wrong**. (Why?)

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Process 1 and 2 have **overlapping outcomes**! For example, 1000 0000.

So the sum rule would **overcount** the outcomes. By how much?

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- **Process 3:** Construct a bit string of length 8 that starts with 1 *and* ends with 00. There are $1 \cdot 2^5 \cdot 1^2 = 32$ outcomes.

We correct for the overcount by subtracting the number of overlapping outcomes:

$$128 + 64 - 32 = 160$$

This technique is called **inclusion-exclusion**.

Inclusion-Exclusion

Inclusion-Exclusion

Let A_1 and A_2 be finite sets, not necessarily disjoint.
Then the number of ways to choose an element from either of the sets is

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Example

$$|\{1,2,3,4\} \cup \{3,4,5\}| = |\{1,2,3,4\}| + |\{3,4,5\}| - |\{3,4\}|$$
$$|\{1,2,3,4,5\}| = 4 + 3 - 2 = 5$$

Permutations and Combinations

Permutations

Example

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Solution:

- There are 26 choices for the first letter.
- There are $26 - 1 = 25$ choices for the second letter, since the first letter must not be reused.
- There are $26 - 2 = 24$ choices for the third letter, since the first and second letters must not be reused.
- There are 10 choices for the first digit.
- There are $10 - 1 = 9$ choices for the second digit, since the first digit must not be reused.

So there are $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 = 1\,404\,000$ total outcomes.

Permutations

Definition (Permutation)

Given a set S, a **permutation** of S is an ordered list whose elements are the elements of S, in some order, without duplicates.

An ***r*-permutation** of S is an ordered list of r elements from S, in some order, without duplicates. (Of course, $0 \le r \le |S|$.)

Example

Let $S = \{1, 2, 3\}$. The permutations (ie, 3-permutations) of S are:

$$(1,2,3) \quad (1,3,2) \quad (2,1,3) \quad (2,3,1) \quad (3,1,2) \quad (3,2,1)$$

The 2-permutations of S are the following:

$$(1,2) \quad (1,3) \quad (2,1) \quad (2,3) \quad (3,1) \quad (3,2)$$

The 1-permutations of S are (1), (2), and (3); and the 0-permutation is $()$.

Counting Permutations

Notation (Permutations)

The number of r-permutations of a set of n elements is written $P(n, r)$.

Suppose a set S has n elements.
How many r-permutations of S are there?
How many ("full") permutations of S are there?

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By the product rule:

$$P(n, r) = \underbrace{n \cdot (n-1) \cdots (n-r+1)}_{r \text{ terms}}$$

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$$P(n,n) = n!$$

Counting Permutations

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Examples

- $P(3,2) = 3!/1! = (3 \cdot 2 \cdot 1)/(1) = 3 \cdot 2 = 3$
- $P(5,3)$
- $P(8,5)$
- $P(10,3)$
- $P(7,0)$

Counting Permutations

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Examples

- $P(3, 2) = 3!/1! = (3 \cdot 2 \cdot 1)/(1) = 3 \cdot 2 = 3$
- $P(5, 3) = 5!/2! = 5 \cdot 4 \cdot 3 = 60$
- $P(8, 5) = 8!/3! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$
- $P(10, 3) = 10!/7! = 10 \cdot 9 \cdot 8 = 720$
- $P(7, 0) = 7!/7! = 1$

Permutations and Combinations

- How many different lists of 3 people can we pick from a set of 8?

  Solution: The outcomes are the 3-permutations of a set of 8.
  So there are $P(8,3) = 8 \cdot 7 \cdot 6 = 336$ outcomes.

- How many different sets of 3 people can we pick from a set of 8?

  Let's start with the solution to the previous problem.
  There are several lists that correspond to each set:

$$
\left.\begin{array}{ccc}(1,2,3) & (2,1,3) & (3,1,2) \\ (1,3,2) & (2,3,1) & (3,2,1)\end{array}\right\} \quad \{1,2,3\}
$$

$$
\left.\begin{array}{ccc}(1,3,5) & (3,1,5) & (5,1,3) \\ (1,5,3) & (3,5,1) & (5,3,1)\end{array}\right\} \quad \{1,3,5\}
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(1,3,2) & (2,3,1) & (3,2,1) & \\
(1,3,5) & (3,1,5) & (5,1,3) & \multirow{2}{*}{$\left.\right\} \quad \{1,3,5\}$} \\
(1,5,3) & (3,5,1) & (5,3,1) & \\
 & & \dots & \quad \dots
\end{array}
$$

  Specifically, there are 6 lists per set, because there are $P(3, 3) = 6$ ways of ordering 3 elements. By the quotient principle, we must *divide* by 6.

  So the answer is $P(8, 3)/P(3, 3) = 336/6 = 56$.

Combinations

Definition (Combination)

Given a set S, an **r-combination** of S is an unordered selection of r elements of S. That is, it is a subset of S with r elements.

Notation (Combinations)

The number of r-combinations of a set with n elements is written $C(n,r)$ or $\binom{n}{r}$, pronounced "n choose r".

Example

Let $S = \{1,2,3,4\}$. The 2-combinations of S are the following:

$$\{1,2\} \quad \{1,3\} \quad \{1,4\} \quad \{2,3\} \quad \{2,4\} \quad \{3,4\}$$

There are 6 of them, so $C(4,2) = 6$.

Counting Combinations

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!\,r!}$$

We can think of the process of generating r-combinations of S as follows:

- First, generate all of the r-permutations of S.

  There are $P(n,r)$ of them, where $n = |S|$.

- Group the lists with the same set of elements together.

  Each set of r elements will appear as $P(r,r)$ different ordered lists.
  That is, we must *divide out* the artificial ordering.

  So the number of distinct sets is $\frac{P(n,r)}{P(r,r)}$.

Counting Combinations

Corollary

Let $n, r \in \mathbb{N}$ with $r \leq n$. Then $C(n,r) = C(n, n-r)$.

Choosing r elements to "take" is the same as choosing $n-r$ elements to "leave".

Proof.

$$C(n,r) = \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,(n-(n-r))!} = C(n, n-r)$$ □

Example

Suppose we have a set S of 6 elements (that is, $n = 6$).

Picking 2 to "take" is essentially the same as picking 4 elements to "leave".

In either case, our number of choices is the number of possibilities to divide the set into one set containing 2 elements and another set containing 4 elements.

Exercise: Permutations or Combinations

Do the following problems involve “permutations” or “combinations”?

- Twelve athletes compete in a footrace. Medals are awarded for first, second, and third place; there are no ties. How many different ways can medals be awarded?
- A soccer club has 8 women and 7 men. For today’s match, the coach wants to field 6 women and 5 men. How many possible teams are there?
- How many sequences of 8 distinct digits are there in which the digits strictly alternate between even and odd?
- How many binary strings of length 7 are there with an even number of ones?
- A license plate consists of 3 distinct letters and 4 distinct numbers, but any position can be occupied by either a letter or a number. How many possible license plates are there?

Exercise: Permutations and Combinations

- A restaurant has an Express Buffet deal. There are 6 dishes offered. You can either choose 3 distinct small servings or 2 distinct medium servings. How many different meals are possible?
- How many sequences of 8 distinct digits (0–9) are there in which the digits strictly alternate between even and odd?
- A concert hall employs 10 ushers. It needs a team of 2 to work the gallery, a team of 4 to work the balcony, and the rest will work the floor. How many different work assignments are there?

Binomial Coefficients

The values of $C(n, k)$ are also called **binomial coefficients**. Why?

- A binomial expression is the sum of two terms, such as $(a + b)$.
- Now consider $(a + b)^k = \underbrace{(a + b)(a + b) \cdots (a + b)}_{k \text{ terms}}$.
- When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor, then we can sum identical terms:

$$\begin{aligned} (a+b)^2 &= aa + ab + ba + bb \\ &= 1a^2 + 2ab + 1b^2 \end{aligned}$$

$$\begin{aligned} (a+b)^3 &= aaa + aab + aba + abb + baa + bab + bba + bbb \\ &= 1a^3 + 3a^2b + 3ab^2 + 1b^3 \end{aligned}$$

There are $C(n, k)$ ways to form a list of k as and $(n - k)$ bs.

Binomial Coefficients

Theorem (The Binomial Theorem)

$$(a+b)^n = \sum_{j=0}^{n} C(n,j)\, a^j\, b^{n-j}$$

With the help of Pascal’s triangle (next section), this formula can considerably simplify the process of expanding powers of binomial expressions.

For example, the fifth row of Pascal’s triangle $(1, 4, 6, 4, 1)$ helps us to compute $(a+b)^4$:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Recurrences

Pascal's Formula

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Let $n, k \in \mathbb{N}$ with $k \leq n$. Then $C(n+1, k) = C(n, k-1) + C(n, k)$.

Why is this true?

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Why is this true?

Suppose $S = \{1, \dots, n+1\}$. How many ways can we choose k elements?

We can focus on the decision whether to include $n+1$:

- If we include $n+1$ in the result,
  we still have to choose $k-1$ elements from $\{1, \dots, n\}$.
  That process has $C(n, k-1)$ outcomes.
- If we do not include $n+1$ in the result,
  we still have to choose k elements from $\{1, \dots, n\}$.
  That process has $C(n, k)$ outcomes.

These are the only two cases, and they do not overlap.

So by the sum rule, $C(n+1, k) = C(n, k-1) + C(n, k)$.

Pascal's Triangle

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

Let's write a "matrix" of the $C(n,k)$ values, where n is the row and k is the column (both indexed starting with 0):

$$\begin{array}{llllll} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \\ \vdots & & & & & \end{array}$$

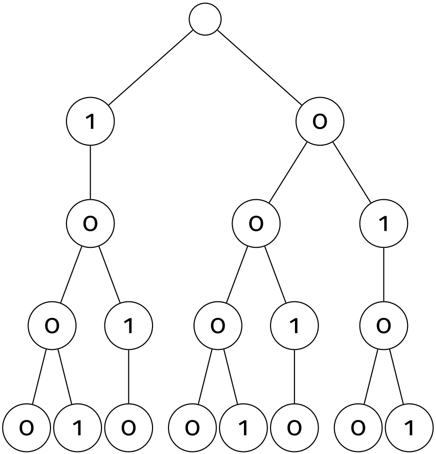
Each number is the sum of the numbers immediately above and above-left.

This is called **Pascal's triangle**.

Recurrences for Counting

Let $S(n)$ be the number of bit strings of length n that do not have two consecutive 1s.

What is $S(n)$?



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Recurrences for Counting

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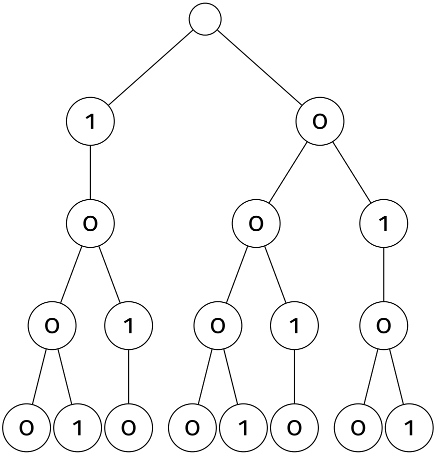
We can make such a bit string of length n in two different ways:

- "0" followed by a no-double-one bit string of length $n-1$
- "10" followed by a no-double-one bit string of length $n-2$

Let's also think about the base cases...



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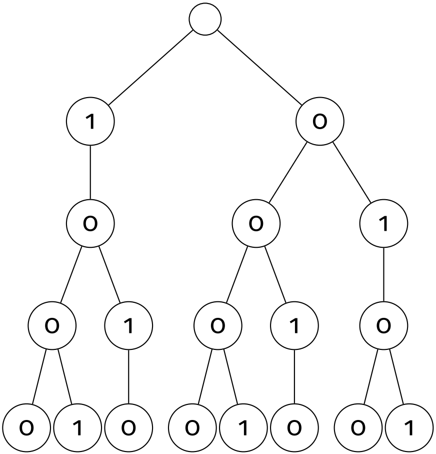
Let's also think about the base cases... Then:

$$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ S(n-1) + S(n-2) & \text{if } n \geq 2 \end{cases}$$



$S(4) = 8$

Pigeonhole Principle

The Pigeonhole Principle

The Pigeonhole Principle

If $k + 1$ or more objects are placed into k boxes, then there is at least one box containing 2 or more of the objects.

Examples

- If there are 11 players on a soccer team that wins 12–0, there must be at least one player in the team who scored at least twice (assuming there are no own-goals!).
- If you have 6 classes schedules Monday to Friday, there must be at least one day on which you have at least two classes.

The Pigeonhole Principle

Example

Assume you have a drawer containing 12 brown socks and 12 black socks all mixed together.

It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

Solution: There are 2 types of socks, so if you pick at least 3 socks, there must be either at least 2 brown socks or at least 2 black socks.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle

In general, if N objects are placed into k boxes, then there is at least one box containing at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Converse of the Generalized Pigeonhole Principle

If there are k boxes, then in order to guarantee that at least one box contains b items, $(b-1)k+1$ items must be distributed among the boxes.

Summary

Summary of Counting Techniques

- **addition (sum rule)** — for counting **"either-or"** situations, where an outcome is produced by either one task or another
  - **subtraction** — for excluding elements
  - **inclusion-exclusion** — for correcting overcounts of overlapping sets
- **multiplication (product rule)** — for counting *"both-and" situations*, where an outcome is created by joining multiple task outputs
  - **quotient** — correct for overcounting by a factor
  - **permutations** — for counting *ordered lists without duplicates* of elements from some set
  - **combinations** — for counting *(unordered) sets (without duplicates)* of elements from some set
- **recursive equations** — (like Pascal's formula)
- **enumeration** — "just count", "tree diagram", etc
- **pigeonhole principle** — for *minimum size of largest group*