Informal Proofs: Guidelines and Examples

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1 Requirements and Guidelines

These are the requirements and guidelines for informal proofs in this class.

- Do not use these symbols: $\therefore, \neg, \land, \lor, \Rightarrow, \Leftrightarrow, \forall, \exists$. Use words instead.

  You may use symbols from set theory and algebra, like $\in, \mathbb{N}, =, \neq, \leq$, etc.
- Use complete sentences. Within a sentence, use a proposition as a clause (as in, "Suppose that $a < b$") or as a noun phrase (as in "Transitivity yields $a < c$").
- Introduce new variables with "let" or "there is" or "some":
  - "let $a, b \in \mathbb{N}$" or "let a and b be natural numbers" — **Let** ($\forall$Intro)
  - "there is some $n \in \mathbb{N}$ such that …" or "… for some $k \in \mathbb{N}$" — $\exists$, $\exists$Elim
  - "there is an $n \in \mathbb{N}$ such that … (in particular, $n =$ *witness*)" — $\exists$Intro
- Introduce assumptions using "Assume" or "Suppose".
- State the sub-goal after introducing variables and/or assumptions in the main body of the proof.
- Cite definitions, axioms, lemmas, and theorems when you use them.

  Skip over sequences of $\forall$Elim, $\Leftrightarrow$Elim{F,B}, and $\Rightarrow$Elim; just state the "result".
  - Omit the variable mapping unless it is especially subtle.
  - Omit the premise arguments (the on parts) if recent and obvious. Otherwise, mention them using "on", "since", "because", etc.
- Introduce chains of equations or inequalities with phrases such as "Consider *expr*:", "Calculate starting with the left side of the goal:", etc.
- (Algebra) Rewrite expressions into the *exact* form required by a definition, axiom, etc.

  Example: $j = 2m^2 + 6m$ does not show j is even, but $j = 2(m^2 + 3m)$ does.
- ($\lor$Elim) Signal the case analysis by saying "case analysis on *condition*".

  Use an itemized or enumerated list for the cases.
- ($\exists$Elim) Use a fresh variable every time you "unpack" an existential.
- (Contradiction) Start with "Assume for the sake of contradiction that *prop*" and end with "From the contradiction, we conclude that *not prop*".
- These phrases are useful for connecting steps of the proof together: "Therefore", "Thus", "So", "Consequently"; "we know", "we get", "we have"; "produces", "yields".

2 Examples with Propositional Logic

2.1 Example: Case Analysis

1 **Axioms:**

2 1. If it is sunny, there is biking and gardening.

3 2. If it is rainy, then there is cleaning.

4 **Theorem.** *If it is rainy or sunny, then there is gardening or cleaning.*

5 *Proof.* Assume that it is rainy or sunny.

6 We want to show there is gardening or cleaning.

7 We proceed by case analysis on the weather:

8 1. *It is rainy.* Then by the second axiom we know there is cleaning.

9 2. *It is sunny.* Then by the first axiom we have biking and gardening.

10 So in either case, we have gardening or cleaning. □

This theorem, like many theorems, is conditional. That is, the main connective is an implication ($\Rightarrow$). The proof starts by assuming the theorem's premise (line 5), and then stating the theorem's conclusion (right-hand side) as the proof's new sub-goal (line 6). This corresponds to the $\Rightarrow$Intro rule. Unlike a formal proof, though, the informal proof concludes immediately after after proving the sub-goal. We could explicitly write "From the hypothetical reasoning above, we conclude that if it is sunny or rainy, then there is gardening or cleaning", but it is usually considered redundant and unnecessary. Note that we are still *following* the $\Rightarrow$Intro rule: the theorem matches $\Rightarrow$Intro's result.

On lines 8–9 I use italics to indicate the two cases. The italicized propositions act as local assumptions, only valid for their branch in the case analysis.

Line 10 summarizes the case analysis by saying that in either case we have gardening or cleaning. In the formal proof, the two cases are required to produce exactly the same result. In the informal version, it is acceptable to eliminate the bookkeeping steps.

2.2 Indirect Proof

Axioms:

1. The crime was committed by a stranger or by a member of the household.
2. If the crime were committed by a stranger, the dog would have barked.

Theorem. *If the dog did not bark, the crime was committed by a member of the household.*

Proof. Assume the dog did not bark.

We want to show that the crime was committed by a member of the household.

From axiom 2 we can deduce that the crime was not committed by a stranger, since the dog did not bark.

By axiom 1, if we eliminate the stranger case, the only remaining possibility is that the crime was committed by a member of the household. □

On lines 5–6, we are applying the Modus Tollens rule, which is like applying $\Rightarrow$Elim to the contrapositive of axiom 2. Instead of talking about the inference rule, though, the informal proof mentions the facts that we are combining and states the conclusion. The reader is trusted to infer the rule being used and check that it is applied correctly.

Similarly, lines 7–8 mention axiom 1 (a disjunction) and indirectly mention the previous result (phrased as "eliminate the stranger case"). The phrasing "eliminate" and "remaining possibility" hint that we are applying Disjunctive Syllogism without spelling it out.

3 Examples with Predicate Logic

3.1 Odd and Even

Here are definitions for even and odd stated in mathematical prose:

> **Definitions.** *A number $n \in \mathbb{N}$ is even iff there is some $k \in \mathbb{N}$ such that $n = 2k$.*
>
> *A number $n \in \mathbb{N}$ is odd iff there is some $k \in \mathbb{N}$ such that $n = 2k + 1$.*
>
> *Every $n \in \mathbb{N}$ is either odd or even, but not both.*

The third line is stronger than the axiom you've seen before, which just says every natural number is either odd or even. Proving that a number cannot be both is difficult and requires additional axioms about the natural numbers.

Here is a basic theorem about even and odd numbers:

1 **Theorem.** *Let $n \in \mathbb{N}$. If n is odd, then $n + 1$ is even.*

2 *Proof.* Let $n \in \mathbb{N}$, and suppose n is odd.

3 We must show that $n + 1$ is even.

4 By the definition of odd, there is some $m \in \mathbb{N}$ such that $n = 2m + 1$.

5 Consider $n + 1$:

6

7 $$n + 1 = (2m + 1) + 1 = 2m + 2 = 2(m + 1) \qquad \text{by algebra}$$

8

9 Thus there is some $k \in \mathbb{N}$ such that $n + 1 = 2k$ (in particular, $k = m + 1$).

10 So $n + 1$ is even, by the definition of even. □

The structure of this theorem is universal with an implication inside, which is probably the most common structure that theorems take. In a formal proof, we would use ∀Intro to handle the universal quantifier and ⇒Intro to handle the implication. The informal proof follows the same structure: line 2 introduces the quantifier-bound variable n using "Let" and then assumes the premise (the implication's left-hand side) with "suppose". Line 3 states the sub-goal: the "result" or conclusion of the theorem: "$n + 1$ is even".

Line 4 says that we are using the definition of odd from the axioms above. Implicitly, we are instantiating that universal definition to talk about *our n variable*, and we are using the definition's "iff" in the *forward* direction. The definition's result, formally, is $\exists k \in \mathbb{N},\ n = 2k$. In an informal proof, however, we generally don't state an existential proposition as an intermediate result. Instead, we automatically "unpack" it with a freshly-chosen variable for the witness. We signal the introduction of the witness variable with a phrase like "there is some $m \in \mathbb{N}$"; that tells the reader that m is a new variable with an unknown value.

Line 5 uses the word "consider" to tell the reader that we are going to start with a particular expression and calculate another expression related to it (in this case, equal to it).

On line 7, note that the final expression is in the form 2 times some quantity, matching the form of the expression in the definition of even *exactly*.

On line 9, we rewrite the previous equation to use a new variable k. We indicate that this equation should be interpreted as an existential proposition by introducing the variable with the phrase "there is some $k \in \mathbb{N}''$. Since we are *creating* an existential statement, we know the value of the witness, and we state it explicitly for clarity: "in particular, $k = m + 1$".

Theorem. *If $n \in \mathbb{N}$ is odd, then n^2 is odd.*

Proof. Let $n \in \mathbb{N}$, and suppose n is odd.

We want to show that n^2 is odd.

Since n is odd, $n = 2k + 1$ for some $k \in \mathbb{N}$.

Consider n^2:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Thus there is an integer m (in particular, $m = 2k^2 + 2k$) such that $n^2 = 2m + 1$. Therefore by definition, n^2 is odd. □

The structure of this proof is similar to the last one. The difference is that we're using odd in both the assumption and the conclusion, and the expression is n^2 instead of $n + 1$, which makes the algebra slightly different.

Theorem. *For every natural number n, if n^2 is even, then n is even.*

Proof. We will prove this by proving the contrapositive.

Let n be a natural number, and assume that n is not even.
We want to show that n^2 is not even.

We know every $n \in \mathbb{N}$ is either even or odd, so if n is not even, then it is odd, and vice-versa.

Since n is not even, it must be odd, so there is some integer k such that $n = 2k + 1$.
Now consider n^2:

$$n^2 \ = \ (2k + 1)^2 \ = \ 4k^2 + 4k + 1 \ = \ 2(2k^2 + 2k) + 1$$

So there is an integer $m = 2k^2 + 2k$ such that $n^2 = 2m + 1$.

Therefore n^2 is odd, and so n^2 is not even.

We have proven that if n is not even, then n^2 is not even, so we can conclude the contrapositive: If n^2 is even, then n is even. □

Proving this "directly" is difficult or impossible. It is easier to prove the contrapositive. But since proving the contrapositive is a relatively uncommon proof technique, it is worth warning the reader that we are doing that rather than following the typical direct strategy (line 2). Then on line 3 we introduce n (following $\forall$Intro) and assume the premise ($\Rightarrow$Intro) of the *contrapositive*. Line 4 states the immediate sub-goal: the conclusion of the contrapositive.

Lines 5–6 essentially state and prove an unnamed lemma: $\forall n \in \mathbb{N},\ \neg\text{Even}(n) \iff \text{Odd}(n)$. We'll use this fact twice, so it makes sense to establish it up front. (A formal proof would use the Disjunctive Syllogism rule here.)

Line 13 achieves the sub-goal established on line 4, the proof of the contrapositive's conclusion. We have already told the reader that we are proving the contrapositive, so we could arguably end the proof at this point. In this case, I prefer to spell it out explicitly instead (lines 14–15).

3.2 Inequalities

The following proofs use these axioms:

Axioms:

1. $\forall a, a', c \in \mathbb{N},\ a \leq a' \ \Rightarrow\ a + c \leq a' + c$
2. $\forall a \in \mathbb{N},\ 0 \leq a$
3. $\forall a, b, c \in \mathbb{N},\ a \leq b \Rightarrow b \leq c \Rightarrow a \leq c$ (transitivity of $\leq$)

Theorem. $\forall m, n \in \mathbb{N},\ m \leq m + n$

Proof. Let $m, n \in \mathbb{N}$.

We want to prove that $m \leq m + n$.

By axiom 2 we have $0 \leq n$. Then we can apply axiom 1 to get

$$0 + m \leq n + m$$

Simplifying and reordering terms, we get

$$m \leq m + n$$

which is what we wanted to show. □

The phrase "which is what we wanted to show" on line 12 is just noise to avoid ending the proof with displayed math.

The following theorem justifies the practice of "adding inequalities":

Theorem. *For all $a, a', b, b' \in \mathbb{N}$, if $a \leq a'$ and $b \leq b'$, then $a + b \leq a' + b'$.*

Proof. Let $a, a', b, b' \in \mathbb{N}$, and suppose $a \leq a'$ and $b \leq b'$.

We want to show that $a + b \leq a' + b'$.

Applying the first axiom with the assumption $a \leq a'$ we get

$$a + b \leq a' + b \tag{1}$$

and likewise, by applying it again with the assumption $b \leq b'$ we get

$$b + a' \leq b' + a' \tag{2}$$

or equivalently (reordering the terms):

$$a' + b \leq a' + b' \tag{3}$$

Then by the transitivity of $\leq$ (axiom 3) applied to the inequalities in (1) and (3), we get

$$a + b \leq a' + b' \tag{4}$$

which is the desired inequality. □

Giving labels to equations and inequalities allows you refer to them later when you use them.

4 Irrationality of $\sqrt{2}$

Definitions. *A number is rational iff it can be expressed as the ratio of two coprime integers. (That is, if it can be written as fraction in "reduced form".)*

That is, $q \in \mathbb{Q}$ iff there exist $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$ such that $q = \frac{a}{b}$ and a and b are coprime.

Two integers are coprime if the have no positive divisor in common greater than 1.

Theorem (circa 450 BC?)**.** *The square root of* 2 *is irrational (that is, it is not expressible as the ratio of two coprime integers).*

Proof. Suppose for the sake of contradiction that $\sqrt{2}$ is rational.

Then let $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}^+$ and a and b are coprime.

Now we square both sides to get $2 = \frac{a^2}{b^2}$. Then we multiply both sides by b^2 to get

$$a^2 = 2b^2 \tag{5}$$

So a^2 is even, and by an earlier theorem, that means a is even; so let $a = 2k$.

Substituting into equation (5), we get $(2k)^2 = 2b^2$, so $4k^2 = 2b^2$, and so

$$b^2 = 2k^2 \tag{6}$$

That means that b^2 is even, so (applying that theorem again), b is even.

But since a and b are both even, they share the positive divisor 2, which contradicts the assumption that a and b are coprime.

Thus we conclude that $\sqrt{2}$ cannot be expressed as the ratio of coprime integers.
That is, $\sqrt{2}$ is irrational. □

5 Induction

5.1 Recursively-Defined Function

Theorem. *Let* $s : \mathbb{N} \to \mathbb{N}$ *be defined by* $s(0) = 0$ *and* $s(n+1) = 2n + 1 + s(n)$. *Then for every* $n \in \mathbb{N}$, $s(n) = n^2$.

Proof. By induction on n.

Base case:

Goal: $s(0) = 0^2$

By definition $s(0) = 0 = 0^2$.

Inductive case:

Let $n \in \mathbb{N}$. Assume (inductive hypothesis): $s(n) = n^2$.

Goal: $s(n+1) = (n+1)^2$.

Calculate starting with $s(n+1)$:

$$\begin{aligned} s(n+1) &= 2n + 1 + s(n) && \text{by definition of } s \\ &= 2n + 1 + n^2 && \text{by IH} \\ &= n^2 + 2n + 1 = (n+1)^2 && \text{algebra} \end{aligned}$$

The chain of equations shows that $s(n+1) = (n+1)^2$, which was the goal. □

The redundant summary on line 14 is to avoid ending the proof with a math display.

5.2 Summations

Theorem (Gauss)**.** *Let* $n \in \mathbb{N}$. *Then* $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof. By induction on n.

Base case:

Goal: $\sum_{k=1}^{0} k = \frac{0\cdot(0+1)}{2}$

Calculate left-hand side: $\sum_{k=1}^{0} k = 0$

Calculate right-hand side: $\frac{0\cdot(0+1)}{2} = \frac{0}{2} = 0$

Both sides of the equation are equal to 0, so the goal is satisfied.

Inductive case:

Let $n \in \mathbb{N}$. Assume (inductive hypothesis): $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Goal: $\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}$.

Let's calculate starting with the left-hand side of the goal:

$$
\begin{aligned}
\sum_{k=1}^{n+1} k &= (n+1) + \sum_{k=1}^{n} k && \text{split the summation} \\
&= (n+1) + \frac{n(n+1)}{2} && \text{by IH} \\
&= \frac{2n+2 \ + \ n^2+n}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}
\end{aligned}
$$

The chain of equations demonstrates the goal. □

Here is another way to prove the inductive case that does not require factoring the quadratic:

Calculate starting with the left-hand side of the goal:

$$
\begin{aligned}
\sum_{k=1}^{n+1} k &= (n+1) + \sum_{k=1}^{n} k && \text{split the summation} \\
&= (n+1) + \frac{n(n+1)}{2} && \text{by IH} \\
&= \frac{2n+2 \ + \ n^2+n}{2} = \frac{n^2+3n+2}{2}
\end{aligned}
$$

Calculate starting with the right-hand side of the goal:

$$
\frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{n^2+3n+2}{2}
$$

The left-hand side and right-hand sides of the goal are both equal to the same expression, so they are equal. □

5.3 Exponents

Theorem. *For all $n \in \mathbb{N}$, $2^n > n$.*

Proof. By induction on n.

Base case:

Goal: $2^0 > 0$

Inductive case:

Let $n \in \mathbb{N}$. Assume (inductive hypothesis): $2^n > n$.

Goal: $2^{n+1} > n + 1$.

Let's calculate starting with the left-hand side of the goal:

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &= 2^n + 2^n \\ &> n + 2^n && \text{by IH} \\ &\geq n + 1 && 2^n \geq 1 \text{ for all } n \in \mathbb{N} \end{aligned}$$

By the chain of inequalities, we get $2^{n+1} > n + 1$. □

5.4 Divisibility and Factorial

Theorem. *Let* $d, n \in \mathbb{N}$*, and suppose* $1 < d \leq n$*. Then* $d \mid (n!)$.

Proof. Let $d \in \mathbb{N}$ and assume $d > 1$.

We must show for every $n \in \mathbb{N}$, if $n \geq d$ then $d \mid (n!)$.

We will show that by induction on n starting at d.

Base case:

Goal: $d \mid (d!)$.

Since $d > 1$, we have $d! = d \cdot (d-1)!$, and $d \mid d \cdot (d-1)!$ by definition, so $d \mid (d!)$.

Inductive case:

Let $n \in \mathbb{N}$. Assume $n \geq d$, and assume (inductive hypothesis): $d \mid (n!)$.

Goal: $d \mid ((n+1)!)$.

Since $d \mid (n!)$, we know there is some $k \in \mathbb{Z}$ such that

$$n! = kd$$

Multiplying both sides by $n+1$ we get

$$(n+1)(n!) = (n+1)kd$$

And then by the definition of factorial:

$$(n+1)! = ((n+1)k)d$$

So there is some $m \in \mathbb{Z}$ (in particular, $m = (n+1)k$) such that $(n+1)! = md$.

Then by definition of divisibility, $d \mid (n+1)!$. □

In this proof, the assumption $n \geq d$ on line 9 is not needed.

5.5 Strong Induction

Theorem. *Every $n \in \mathbb{N}$ greater than 1 can be written as a product of (one or more) primes.*

Proof. By strong induction on n starting at 2.

Base case:

Goal: 2 is a product of primes.

2 is prime, so the product of primes is just 2.

Inductive case:

Let $n \in \mathbb{N}$. Assume (inductive hypothesis): for every $k \leq n$, k is a product of primes

Goal: $n + 1$ is a product of primes

By case analysis on whether $n + 1$ is prime or composite:

- If $n + 1$ is prime, then the product of primes is just $n + 1$.
- If $n+1$ is composite, then there are $a, b \in \mathbb{N}$ such that $n+1 = ab$ and $1 < a < n+1$ and $1 < b < n + 1$.

  From these inequalities we can get $a \leq n$ and $b \leq n$, so we can apply the induction hypothesis to get a product of primes for a and a product of primes for b.

  Multiplying them together gives a product of primes for $n + 1$, as required.

In either case, $n + 1$ is expressible as a product of primes. □

6 Structural Induction

6.1 Binary Trees

Let BT be defined as follows:

> Let BT be the smallest set such that
>
> - $\textsc{nil} \in BT$, and
> - $\textsc{node}(n, t_l, t_r) \in BT$ for every $n \in \mathbb{Z}$ and $t_l, t_r \in BT$

Let $count : BT \to \mathbb{N}$ and $height : BT \to \mathbb{N}$ be defined as follows:

$$\begin{aligned} count(\textsc{nil}) &= 0 \\ count(\textsc{node}(n, t_l, t_r)) &= 1 + count(t_l) + count(t_r) \\[4pt] height(\textsc{nil}) &= 0 \\ height(\textsc{node}(n, t_l, t_r)) &= 1 + \max(height(t_l), height(t_r)) \end{aligned}$$

Here is a theorem that relates them:

> **Theorem.** *For every $t \in BT$, $count(t) \geq height(t)$.*
>
> *Proof.* By structural induction on t.
>
> **Base case:** $P(\textsc{nil})$
>
> Goal: $count(\textsc{nil}) \geq height(\textsc{nil})$.
>
> Both sides evaluate to 0.
>
> **Inductive case:** $\forall x \in \mathbb{Z},\ \forall t_l, t_r \in BT,\ (P(t_l) \land P(t_r)) \Rightarrow P(\textsc{node}(x, t_l, t_r))$
>
> Let $x \in \mathbb{Z}$, and let $t_l, t_r \in BT$.
>
> Assume (inductive hypotheses): $count(t_l) \geq height(t_l)$ and $count(t_r) \geq height(t_r)$.
>
> Goal: $count(\textsc{node}(x, t_l, t_r)) \geq height(\textsc{node}(x, t_l, t_r))$.
>
> Calculate starting with the left-hand side:
>
> $$\begin{aligned} count(\textsc{node}(x, t_l, t_r)) &= 1 + count(t_l) + count(t_r) && \text{by definition of } count \\ &\geq 1 + height(t_l) + height(t_r) && \text{by IH} \times 2 \\ &\geq 1 + \max(height(t_l), height(t_r)) && \forall a, b \in \mathbb{N},\ a + b \geq \max(a, b) \\ &= height(\textsc{node}(x, t_l, t_r)) && \text{by definition of } height \end{aligned}$$
>
> So the desired inequality is shown. □

The definition of BT has one base case (NIL), so a structural induction proof has one base case, where we must prove the property in question holds for NIL. The definition of BT has one recursive case, which draws *two* variables from BT to use to construct the NODE to add to the set. So a structural induction proof has an inductive case with *two* variables and *two* inductive hypotheses: we assume the property already holds for those two variables. Then we must prove that it holds for the "new" element added to the set, the NODE value.

(The text in red is not part of the proof.)

6.2 More Structural Induction (1)

Let the set E be defined as follows:

Let $E \subseteq \mathbb{Z}$ be the smallest set such that

- $2 \in E$
- if $m, n \in E$, then $m - n \in E$

In "ordinary" induction (on $\mathbb{N}$), the inductive case relies on having already shown the property for *numerically smaller* elements of $\mathbb{N}$. That won't work here, because E is constructed "out of order".

The definition of E has one base case: 2. So a structural induction proof on E has one base case: we must prove the property holds for 2.

The definition of E has one recursive case, and that recursive case draws *two* elements from E to use to form the "new" element to "add" to E. So a structural induction proof on E has one inductive case, but that inductive case has *two* induction hypotheses. We assume that the property holds of both elements drawn from previous rounds, and then we must prove that the property holds for the "new" element constructed in the recursive case.

Theorem. *Every $n \in E$ is even.*

Proof. By structural induction on n.

Base case: $P(2)$

Goal: 2 is even.

By definition of even, since $2 = 2 \cdot 1$.

Inductive case: $\forall m, n \in E,\ (P(m) \land P(n)) \Rightarrow P(m - n)$

Let $m, n \in E$.

Assume (inductive hypotheses): m is even and n is even.

Goal: $m - n$ is even.

Since m is even, $m = 2a$ for some $a \in \mathbb{Z}$.

Since n is even, $n = 2b$ for some $b \in \mathbb{Z}$.

Then $m - n = 2a - 2b = 2(a - b)$, so $m - n$ is even. $\square$

(The text in red is not part of the proof.)

6.3 More Structural Induction (2)

Let the set X be defined as follows:

> Let $X \subseteq \mathbb{R}$ be the smallest set such that
> - $1 \in X$
> - if $x, y \in X$, then $x + y \in X$
> - if $x \in X$ and $x \neq 0$, then $1/x \in X$

The recursive definition of X has one base case and two recursive cases. So a structural induction proof will have one base case and two inductive cases. The first inductive case draws two variables from X so there are two inductive hypotheses, one for each variable. The second inductive case draws a single variable from X but adds a condition, so the second inductive case has one inductive hypothesis but also adds the condition as an assumption.

> **Theorem.** *Every $z \in X$ is positive.*
>
> *Proof.* By structural induction on z.
>
> **Base case:** $P(1)$
>
> Goal: 1 is positive.
>
> Yes, 1 is positive.
>
> **Inductive case 1:** $\forall x, y \in X,\ (P(X) \wedge P(y)) \Rightarrow P(x+y)$
>
> Let $x, y \in X$. Assume (inductive hypotheses): x is positive, y is positive.
>
> Goal: $x + y$ is positive.
>
> If x and y are both positive, then $x + y$ is also positive.
>
> **Inductive case 2:** $\forall x \in X,\ x \neq 0 \Rightarrow P(X) \Rightarrow P(1/x)$
>
> Let $x \in X$. Assume $x \neq 0$ and assume (inductive hypothesis): x is positive.
>
> Goal: $1/x$ is positive.
>
> If x is positive, then $1/x$ is also positive. □

(The text in red is not part of the proof.)