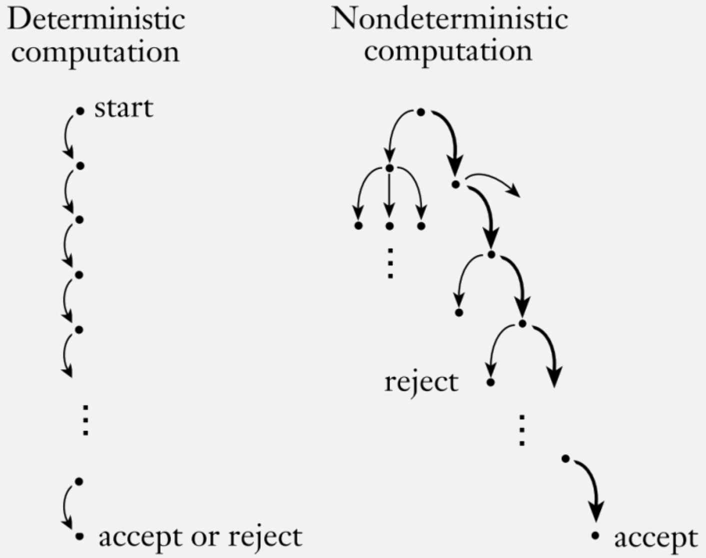
CS 420

Nondeterminism

Thursday, September 22, 2022

UMass Boston Computer Science



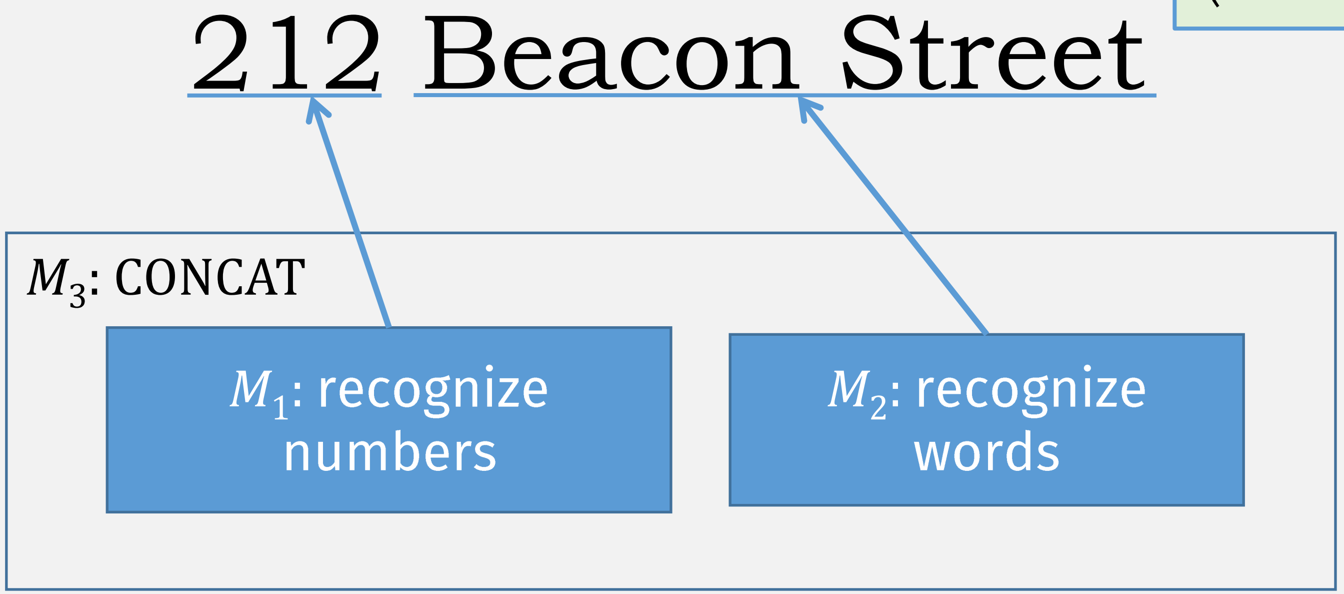
Announcements

- HW 1
  - due Sun Sept 25 11:59pm EST

Last Time: Concatenation

We want this operation
to be **closed** ...
allows using DFAs as
building blocks
(~ modular programming)

Example: **Recognizing street addresses**



Last Time: Is Concatenation Closed?

FALSE?

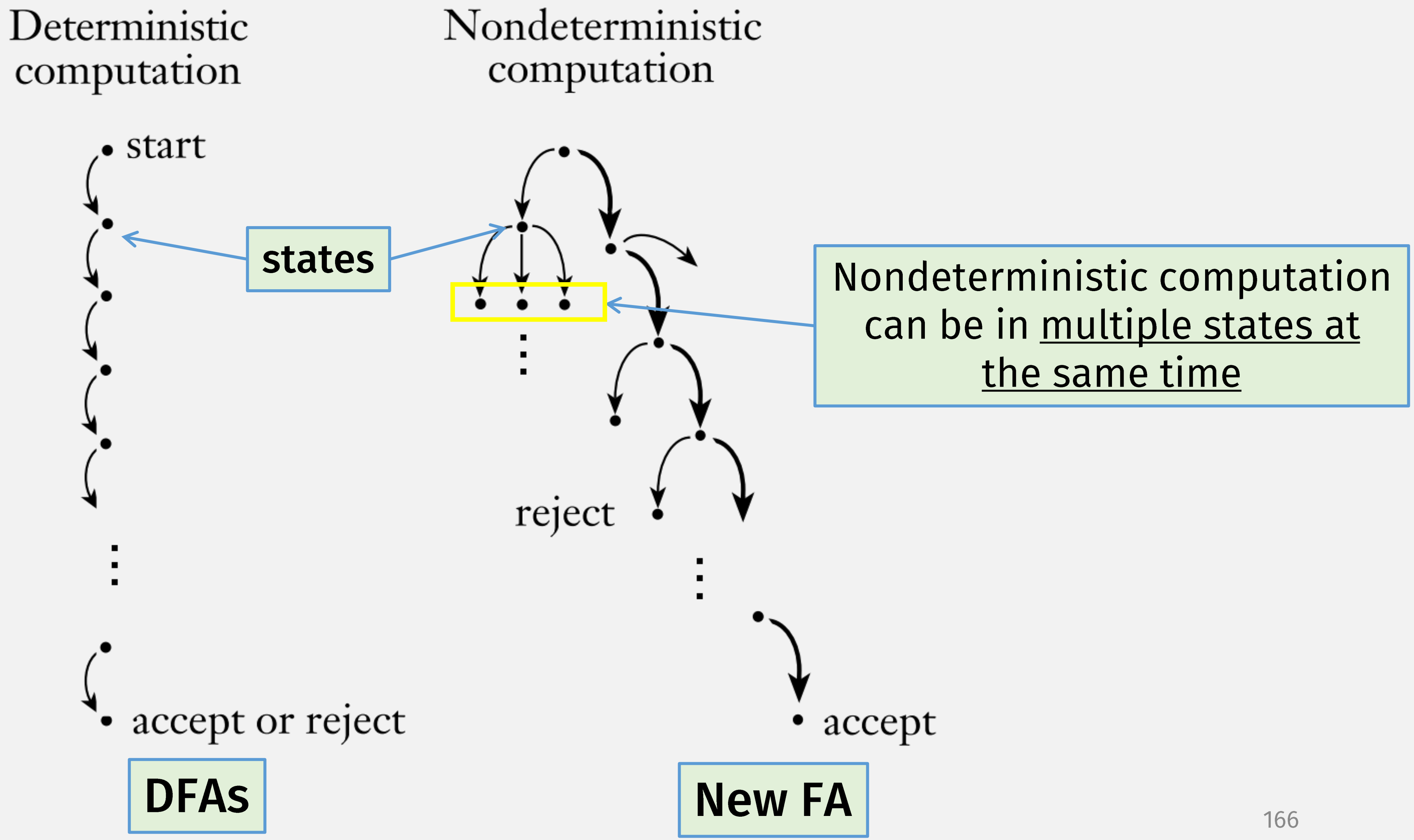
THEOREM

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

- Cannot **combine A_1 and A_2's machine because:**
  - **Not clear when to switch machines?** (can only read input once)
- **Requires a <u>new kind of machine</u>!**
- <u>But</u> **does this mean concatenation <u>is not closed</u> for regular langs?**

Last Time: Deterministic vs Nondeterministic



Last Time: Formal Definition of NFAs

DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Compare with DFA:

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta: Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

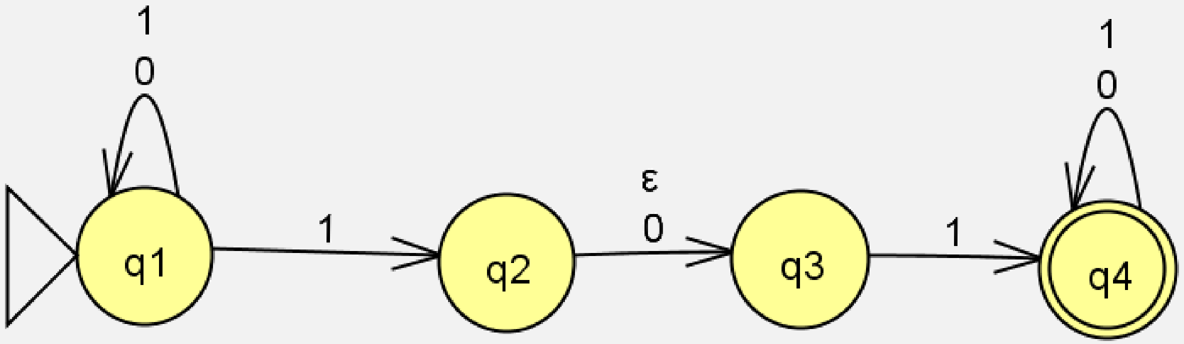
Different transition fn

NFA transition may not read input, $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Transition results in a <u>set of states</u>

Last Time: NFA Example

- Come up with a formal description of the following NFA:



DEFINITION

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5. $F \subseteq Q$ is the set of accept states.

The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

$$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$$

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

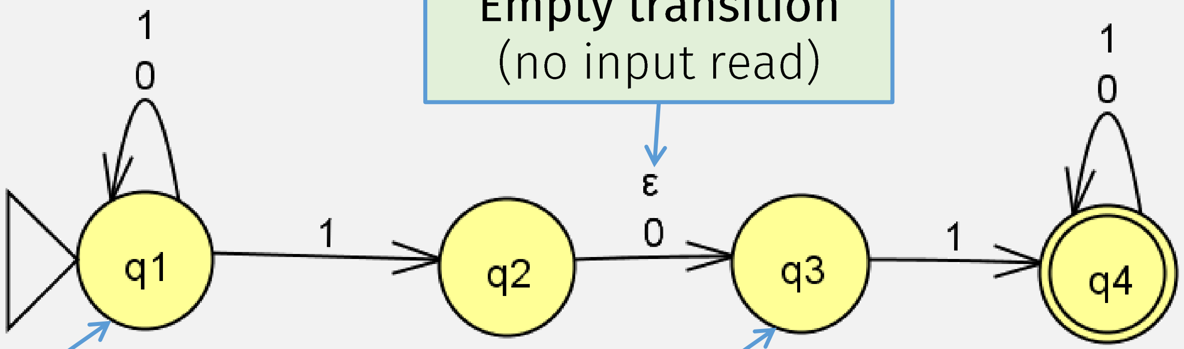
Empty transition (no input read)

Result of transition is a set

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

4. q_1 is the start state, and
5. $F = \{q_4\}$.

Empty transition (no input read)

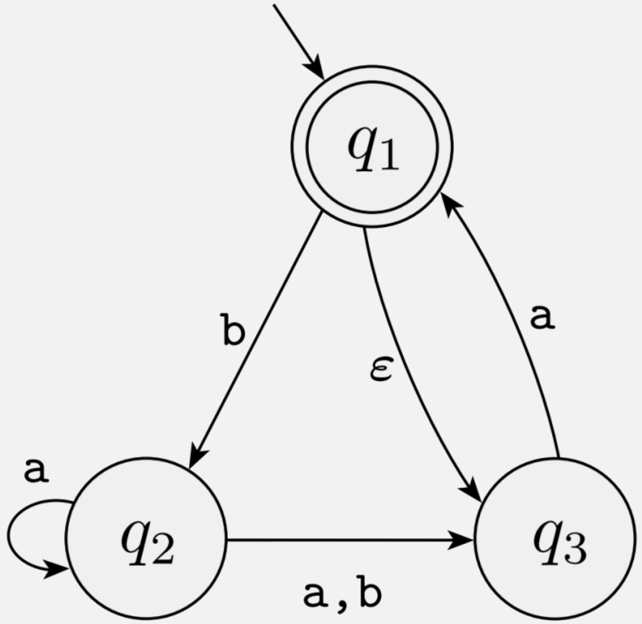


Multiple 1 transitions

No 0 transition

In-class Exercise

- Come up with a formal description for the following NFA
  - $\Sigma = \{\texttt{a}, \texttt{b}\}$



DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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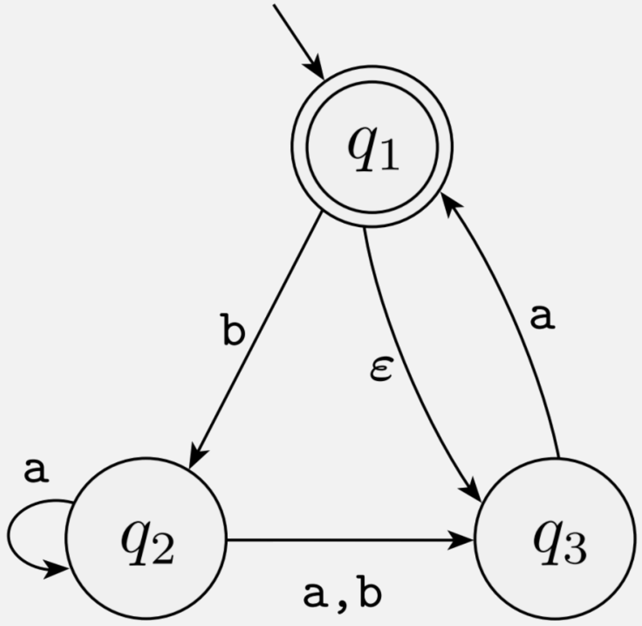
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

In-class Exercise Solution

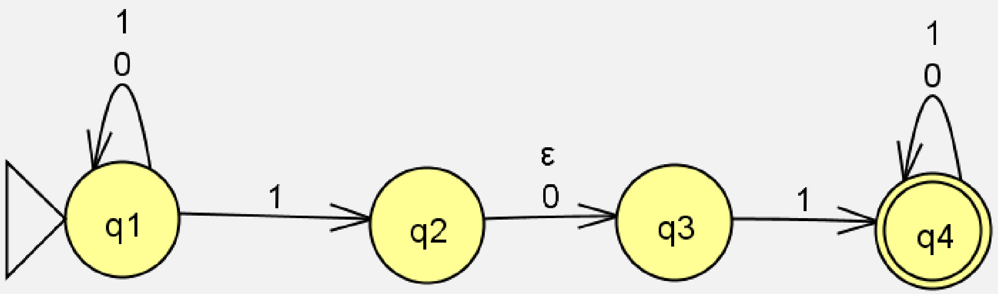
Let $N = (Q, \Sigma, \delta, q_0, F)$

- $Q = \{ q_1, q_2, q_3 \}$
- $\Sigma = \{ \texttt{a}, \texttt{b} \}$
- δ …
- $q_0 = q_1$
- $F = \{ q_1 \}$

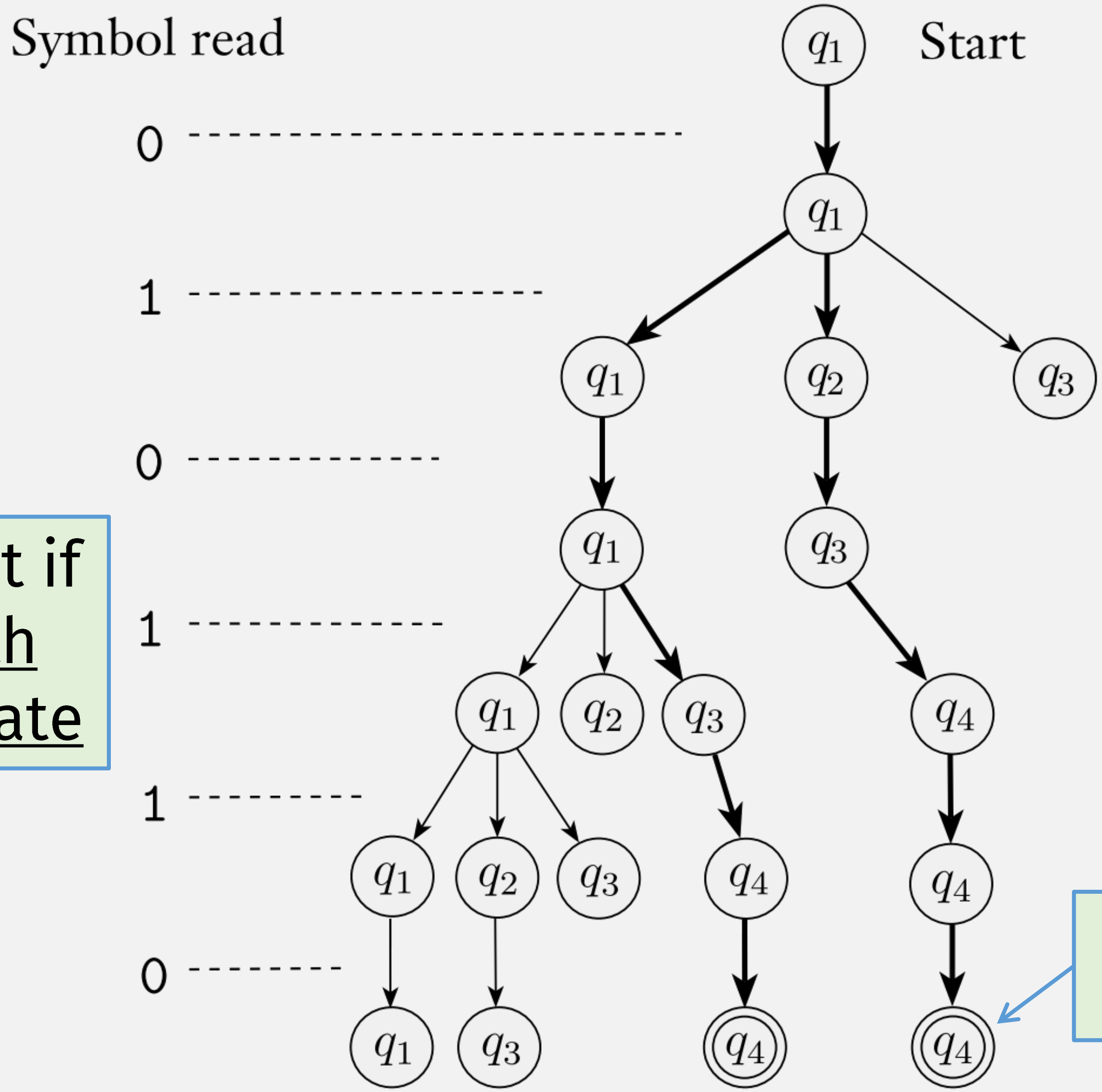
$\delta(q_1, \texttt{a}) = \{ \}$
$\delta(q_1, \texttt{b}) = \{ q_2 \}$
$\delta(q_1, \varepsilon) = \{ q_3 \}$
$\delta(q_2, \texttt{a}) = \{ q_2, q_3 \}$
$\delta(q_2, \texttt{b}) = \{ q_3 \}$
$\delta(q_2, \varepsilon) = \{ \}$
$\delta(q_3, \texttt{a}) = \{ q_1 \}$
$\delta(q_3, \texttt{b}) = \{ \}$
$\delta(q_3, \varepsilon) = \{ \}$



NFA Computation (JFLAP demo): `010110`



NFA Computation Sequence



NFA **accepts** input if at least <u>one path ends in accept state</u>

Each step can branch into <u>multiple states at the same time</u>!

So this is an **accepting computation**

Flashback: DFA Computation Model

Informally

- Machine = a DFA
- Input = string of chars, e.g. "`1101`"

Machine "accepts" input if:

- Start in "start state"
- Repeat:
  - Read 1 char;
  - Change state according to the transition table
- Result =
  - Last state is "Accept" state

Formally (i.e., mathematically)

- $M = (Q, \Sigma, \delta, q_0, F)$
- $w = w_1 w_2 \cdots w_n$

M ***accepts*** w if

sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with

- $r_0 = q_0$
- $r_i = \delta(r_{i-1}, w_i)$, for $i = 1, \ldots, n$
- $r_n \in F$

NFA

~~*Flashback:* DFA~~ Computation Model

Informally	*Formally (i.e., mathematically)*

- Machine = ~~a DFA~~ an NFA
- Input = string of chars, e.g. “`1101`”

Machine “accepts” input if:

- Start in “start state”
- Repeat:
  - Read 1 char;
  - Change state according to the transition table
- Result =
  - Last state is “Accept” state

- $M = (Q, \Sigma, \delta, q_0, F)$
- $w = w_1 w_2 \cdots w_n$

M ***accepts*** w if

sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with

- $r_0 = q_0$
- $\cancel{r_i = \delta(r_{i-1}, w_i)}$, for $i = 1, \ldots, n$

  $r_i \in \delta(r_{i-1}, w_i)$

This is now a set

- $r_n \in F$

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

$\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***

Flashback: DFA Extended Transition Function

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \to Q$

Domain:

- Beginning state $q \in Q$ (not necessarily the start state)
- Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$

Range:

- Ending state (not necessarily an accept state)

(Defined recursively, on length of input string)

- Base case: $\hat{\delta}(q, \varepsilon) = q$
- Recursive case: $\hat{\delta}(q, w) = \hat{\delta}(\delta(q, w_1), w_2 \cdots w_n)$

Empty string

nonEmpty string

First char

Remaining chars
("smaller argument")

Recursive call

Single transition step

$\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***

Alternate Extended Transition Function

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \rightarrow Q$

Domain:

- Beginning state $q \in Q$ (not necessarily the start state)
- Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$

Range:

- Ending state (not necessarily an accept state)

(Defined recursively, on length of input string)

Empty string

- Base case: $\hat{\delta}(q, \varepsilon) = q$

nonEmpty string

Recursive call: (smaller argument) computation "so far"

$\delta(\hat{\delta}(q, w_1 \cdots w_{n-1}), w_n)$

- Recursive case: $\hat{\delta}(q, w) = \cancel{\hat{\delta}(\delta(q, w_1), w_2 \cdots w_n)}$

Single transition step, on last char

NFA

Extended Transition Function

Define **extended transition function:** $\cancel{\hat{\delta} : Q \times \Sigma^* \rightarrow Q}$

$$\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$$

Domain:

- Beginning state $q \in Q$
- Input string $w = w_1 w_2 \cdots w_n \quad \text{where } w_i \in \Sigma$

Range:

- Ending ~~state~~ set of states

Result is set of states

NFA Extended Transition Function

Define **extended transition function:** ~~$\hat{\delta} : Q \times \Sigma^* \rightarrow Q$~~

$\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$

Domain:

- Beginning state $q \in Q$
- Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$

Range:

- Ending ~~state~~ set of states

Result is set of states

(Defined recursively, on length of input string)

Empty string

- Base case: $\hat{\delta}(q, \epsilon) = \{q\}$

nonEmpty string

All single transition steps for last char

- Recursive case: $\hat{\delta}(q, w) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$

Recursive call: (smaller argument) computation "so far"

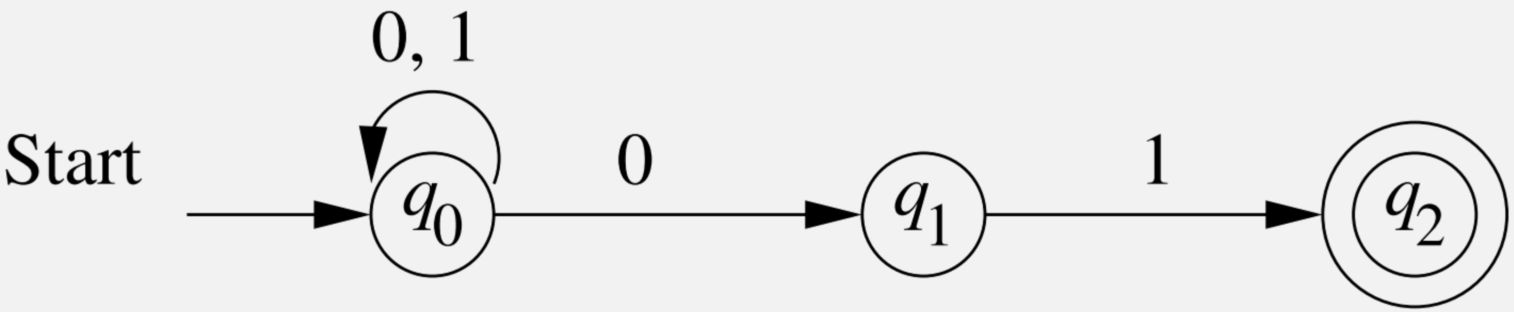
where: $\hat{\delta}(q, w_1 \cdots w_{n-1}) = \{q_1, \ldots, q_k\}$

NFA Extended δ Example

Base case: $\hat{\delta}(q, \epsilon) = \{q\}$

Recursive case: $\hat{\delta}(q, w) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$

where:
$\hat{\delta}(q, w_1 \cdots w_{n-1}) = \{q_1, \ldots, q_k\}$



- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$

Stay in start state

We haven't considered **empty transitions!**

- $\hat{\delta}(q_0, 0) = \delta(q_0, 0) = \{q_0, q_1\}$

Same as single step δ

Combine result of recursive call with "last step"

- $\hat{\delta}(q_0, 00) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 001) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$

Adding Empty Transitions

- Define **the set** $\varepsilon\text{-REACHABLE}(q)$
  - ... to be all states reachable from q via zero or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

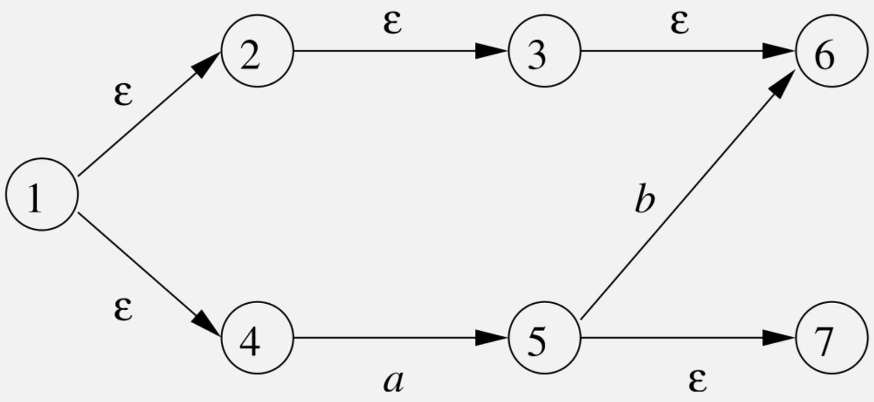
- Inductive case:

A state is in the reachable set if ...

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

... there is an empty transition to it from another state in the reachable set

ε-REACHABLE Example



$$\varepsilon\text{-REACHABLE}(1) = \{1, 2, 3, 4, 6\}$$

No empty transitions

NFA Extended Transition Function

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$

Domain:

- Beginning state $q \in Q$
- Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$

Range:

- Ending set of states

(Defined recursively, on length of input string)

- Base case: $\hat{\delta}(q, \epsilon) = \{q\}$
- Recursive case: $\hat{\delta}(q, w) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$ where: $\hat{\delta}(q, w_1 \cdots w_{n-1}) = \{q_1, \ldots, q_k\}$

With empty transitions

NFA Extended Transition Function

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$

Domain:

- Beginning state $q \in Q$
- Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$

Range:

- Ending set of states

(Defined recursively, on length of input string)

"Take single step,
then follow all empty transitions"

- Base case: $\hat{\delta}(q, \epsilon) = \cancel{\{q\}}$ $\varepsilon\text{-REACHABLE}(q)$
- Recursive case: $\hat{\delta}(q, w) = \cancel{\bigcup_{i=1}^{k} \delta(q_i, w_n)}$ $\varepsilon\text{-REACHABLE}(\bigcup_{i=1}^{k} \delta(q_i, w_n))$

where: $\hat{\delta}(q, w_1 \cdots w_{n-1}) = \{q_1, \ldots, q_k\}$

Summary: NFAs vs DFAs

DFAs

- Can only be in one state
- Transition:
  - Must read 1 char
- Acceptance:
  - If final state is accept state

NFAs

- Can be in multiple states
- Transition
  - Can read no chars
  - i.e., empty transition
- Acceptance:
  - If one of final states is accept state

Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Last Time: Concatenation of Languages

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

If $A = \{\mathtt{good}, \mathtt{bad}\}$ and $B = \{\mathtt{boy}, \mathtt{girl}\}$, then

$$A \circ B = \{\mathtt{goodboy}, \mathtt{goodgirl}, \mathtt{badboy}, \mathtt{badgirl}\}$$

Concatenation: $A \circ B = \{xy | x \in A \text{ and } y \in B\}$

Last Time: Concatenation is Closed?

THEOREM

The class of regular languages is closed under the concatenation operation.

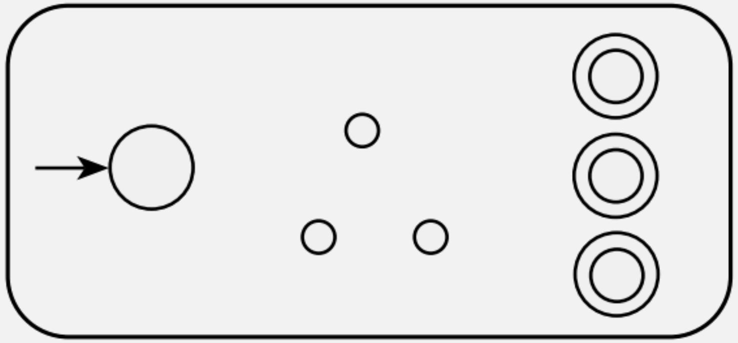
In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Proof: Construct a new machine

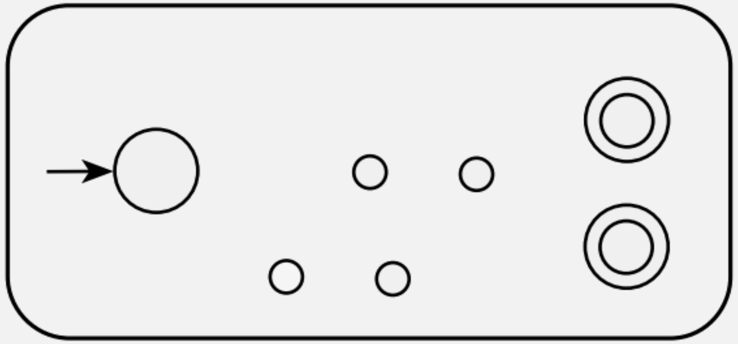
- How does it know when to switch machines?
  - Can only read input once

Concatentation

N_1



N_2



Let N_1 recognize A_1, and N_2 recognize A_2.

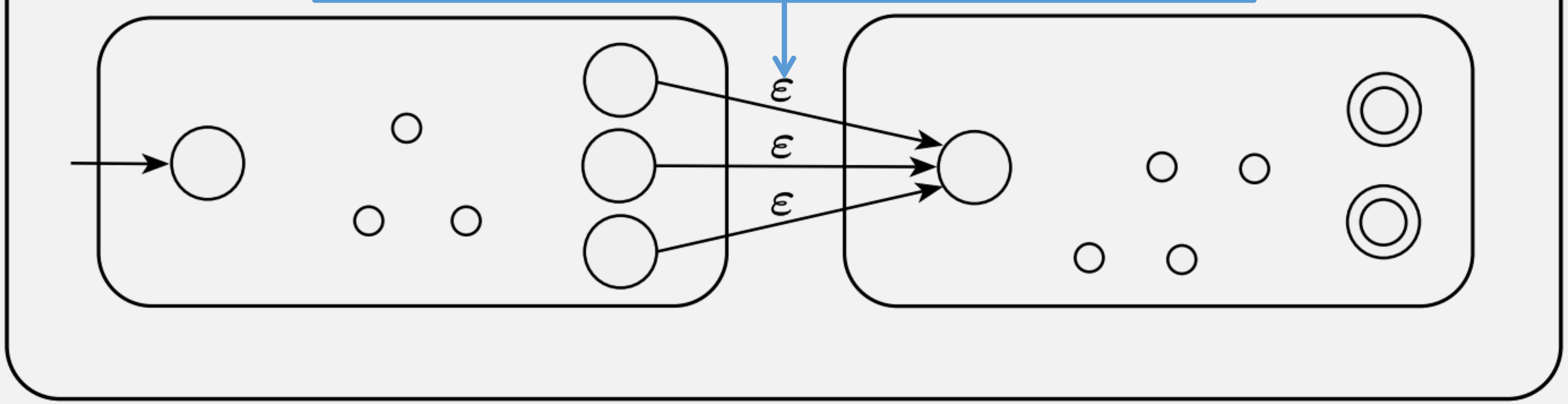
Want: Construction of N to recognize $A_1 \circ A_2$

N is an NFA! It simultaneously:
- Keeps checking 1st part with N_1

  and
- Moves to N_2 to check 2nd part

N

ε = "empty transition" = reads no input
Allows N to be in **both machines at once**



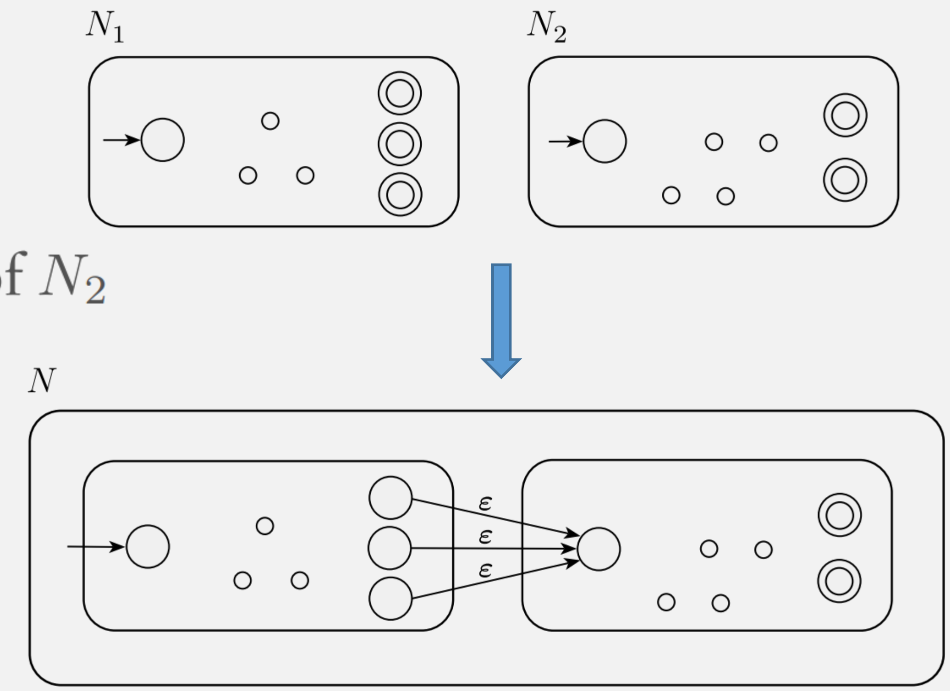
Concatenation is Closed for Regular Langs

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of N_1
3. The accept states F_2 are the same as the accept states of N_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,



Concatenation is Closed for $\boxed{\text{Regular Langs}}$

Wait, is this true?

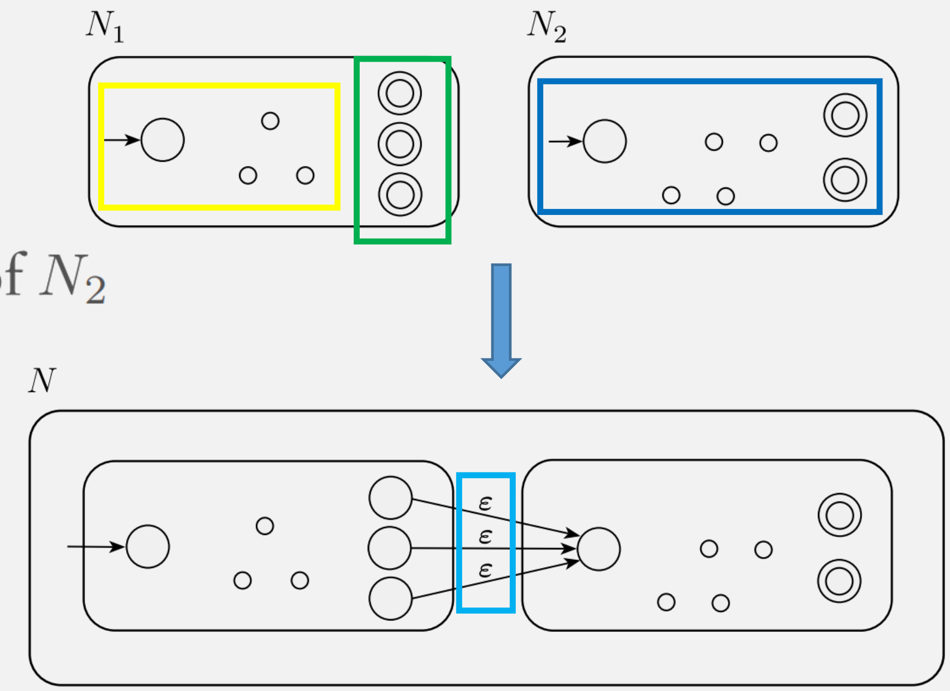
PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of N_1
3. The accept states F_2 are the same as the accept states of N_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(?, a) & \boxed{q \in Q_1 \text{ and } q \notin F_1} \\ \delta_1(?, a) & \boxed{q \in F_1 \text{ and } a \neq \varepsilon} \\ \delta_1(?, a) \cup \{q_2\} & q \in F_1 \text{ and } \boxed{a = \varepsilon} \\ \delta_2(?, a) & \boxed{q \in Q_2.} \end{cases}$$



??? ∎

Flashback: A DFA's Language

- For DFA $M = (Q, \Sigma, \delta, q_0, F)$

- M ***accepts*** w if $\hat{\delta}(q_0, w) \in F$

- M ***recognizes language*** A if $A = \{w \mid M \text{ accepts } w\}$

- A language is a **regular language** if a DFA recognizes it

An NFA's Language

- For NFA $N = (Q, \Sigma, \delta, q_0, F)$

intersection

accept states

- N ***accepts*** w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$ not empty
    - i.e., if the final states have at least one accept state

- Language of N = $L(N)$ = $\left\{ w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}$

Q: How does an NFA's language relate to regular languages

- Definition: A language is regular if a **DFA** recognizes it

Is Concatenation Closed for Reg Langs?

- Concatenation of DFAs produces an NFA

To finish the proof …

- we must <u>prove</u> that NFAs *also* recognize regular languages.

Specifically, we must <u>prove</u>:

- $\boxed{\text{NFAs} \Leftrightarrow \text{regular languages}}$

Check-in Quiz 9/22

On gradescope