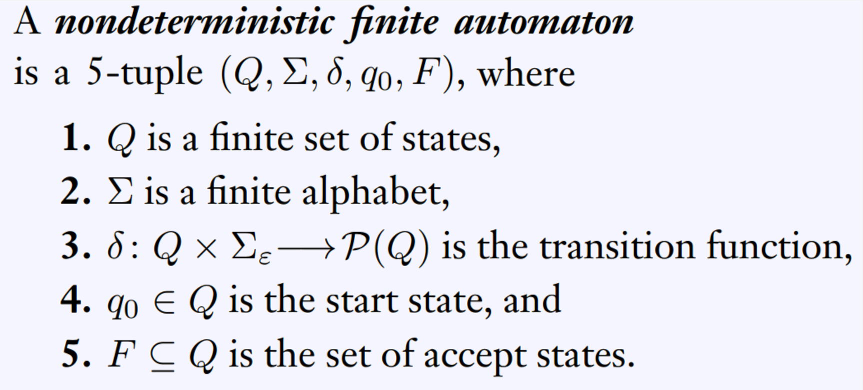
CS420

NFA -> DFA

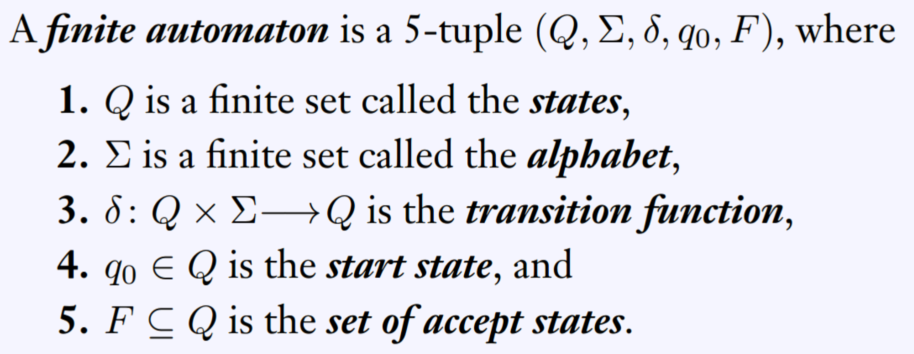
Tuesday, September 27, 2022

UMass Boston CS



A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_{\varepsilon} \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.



A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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5. $F \subseteq Q$ is the ***set of accept states***.

Announcements

- HW 1 in
  - ~~Due Sun 10/25 11:59pm EST~~
- HW 2 out
  - Due Sun 10/2 11:59pm EST
- Ask HW questions publicly on Piazza
  - So the entire class can participate and benefit from the discussion
  - (Make it anonymous if you want to)
- Recipe: **Designing a machine = programming**
  - Make examples to understand problem
  - States = what the machine needs to remember
  - Check design with tests

Flashback: Kinds of Mathematical Proof

Deductive Proof

- Start with known facts and statements
- Use logical **inference rules** to reach new **conclusions**

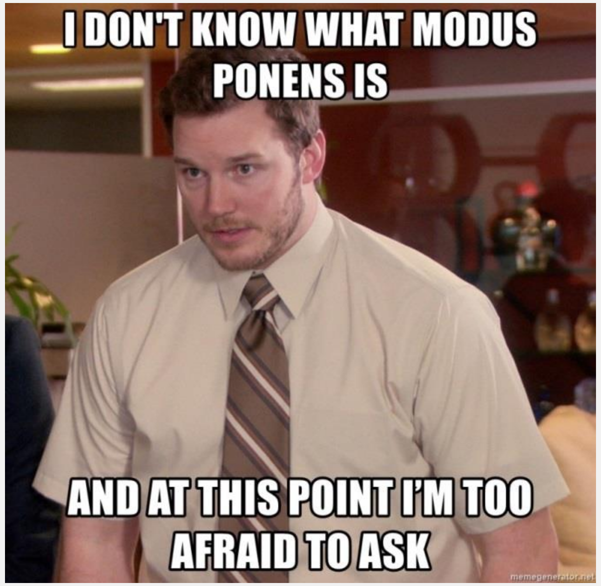
An (Important) Inference Rule: Modus Ponens

Premises

- If P then Q
- P is true

Conclusion

- Q must also be true



Deductive Proof Example

Prove the following:

- If: If $x \geq 4$, then $2^x \geq x^2$
- And: x is the sum of the squares of four positive integers
- Then: $2^x \geq x^2$

Given

Need to show this

Deductive Proof Example

Prove: If If $x \geq 4$, then $2^x \geq x^2$ and x is the sum of the squares of four positive integers
then $2^x \geq x^2$

Proof:

Statement	**Justification**
1. $x = a^2 + b^2 + c^2 + d^2$	1. Given
2. $a \geq 1;\ b \geq 1;\ c \geq 1;\ d \geq 1$	2. Given
3. $a^2 \geq 1;\ b^2 \geq 1;\ c^2 \geq 1;\ d^2 \geq 1$	3. By Step (1) & arithmetic laws
4. $x \geq 4$	4. (1), (3), and arithmetic
5. If $x \geq 4$, then $2^x \geq x^2$	5. Given
6. $2^x \geq x^2$	6. (4) and (5)

■

Deductive Proof Example: Regular Lang

Prove: The following language $A = \{ \dots \}$ is a regular language

Proof:

Statement	**Justification**
1. If a DFA recognizes a language, then that language is regular	1. Definition of a regular language
2. DFA $M = (Q, \Sigma, \delta, q_{\text{start}}, F)$ where Q = …, etc., recognizes language A	2. Definition of a DFA and DFA computation rule
3. A is a regular language	3. By Steps (1) and (2)

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Deductive Proof Example: Closed Op?

Prove: The operation $OP = \{ \ldots \}$ is closed for regular languages

Proof:

Statement	**Justification**
1. ???	1. ???
• ???	• ???
• OP is closed for regular languages	• ???

Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Last Time: Concatenation of Languages

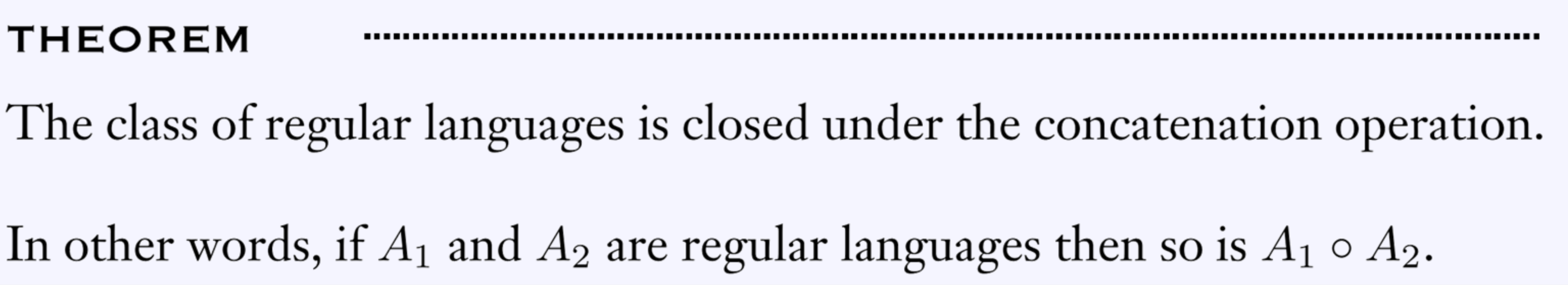
Let the alphabet Σ be the standard 26 letters $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$.

If $A = \{\texttt{good}, \texttt{bad}\}$ and $B = \{\texttt{boy}, \texttt{girl}\}$, then

$$A \circ B = \{\texttt{goodboy}, \texttt{goodgirl}, \texttt{badboy}, \texttt{badgirl}\}$$

Last Time: Is Concatenation Closed?

FALSE?



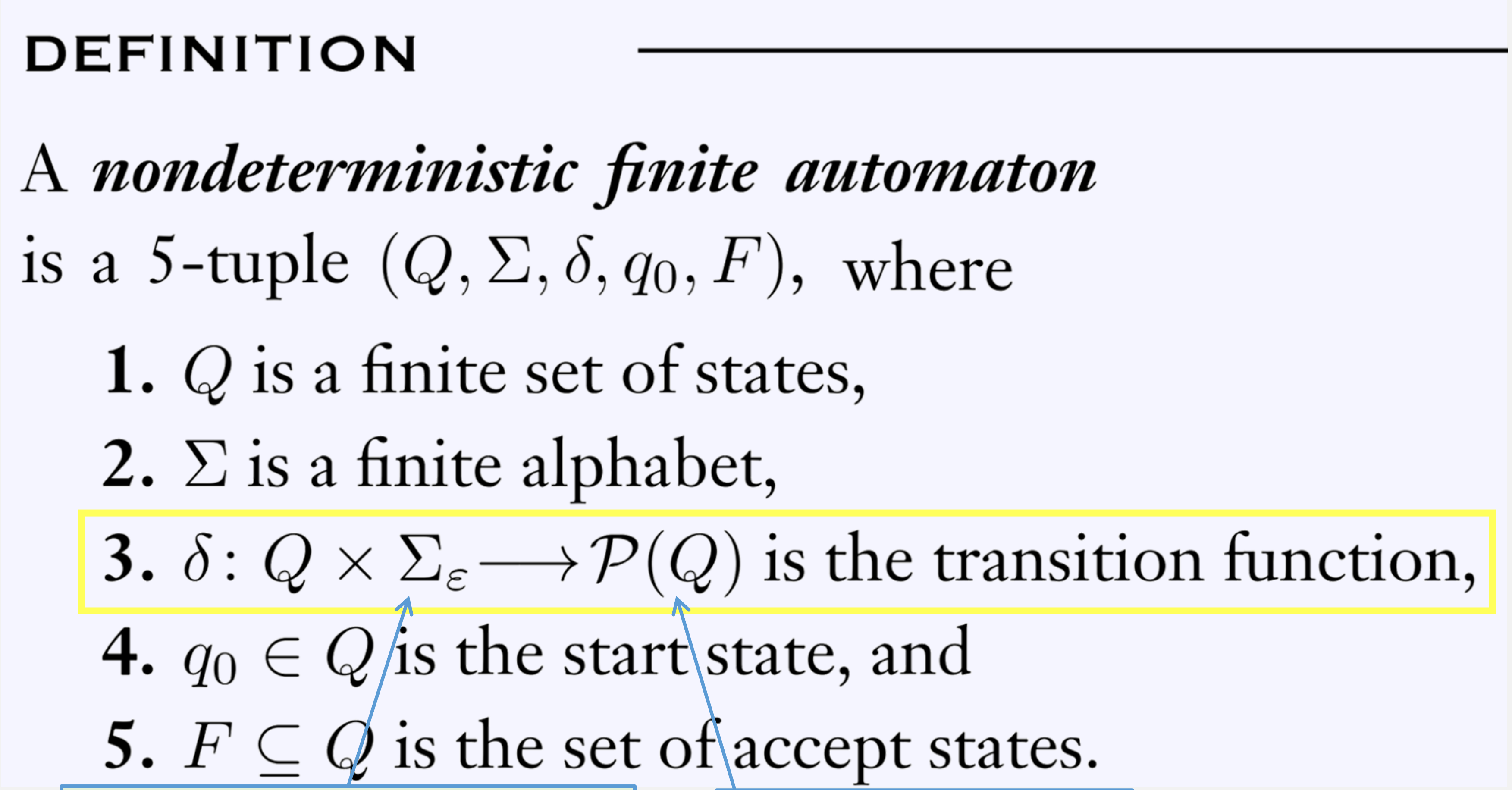
THEOREM

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

- Cannot **combine A_1 and A_2's machine because:**
  - **Don't know when to switch?** (can only read input once)
- **Need a <u>different machine</u>!**
- So **concatenation <u>not closed</u> for regular langs?**

Last Time: NFA Formal Definition



DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

NFA transition may not read input, $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Transition results in a <u>set of states</u>

No empty transitions

Last Time: NFA Extended Transition Function

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$

Transition results in a set of states

Domain:
- Beginning state $q \in Q$
- Input string $w = w_1 w_2 \cdots w_n \quad \text{where } w_i \in \Sigma$

Range:
- Ending set of states

(Defined recursively, on length of input string)

- Base case: $\hat{\delta}(q, \epsilon) = \{q\}$

Combine last single steps for last char

Current states, right before last step

- Recursive case: $\hat{\delta}(q, w) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$ $\quad \text{where: } \hat{\delta}(q, w_1 \cdots w_{n-1}) = \{q_1, \ldots, q_k\}$

Last Time: Adding Empty Transitions

- Define **the set** $\varepsilon\text{-REACHABLE}(q)$
  - … to be all states reachable from q via zero or more empty transitions

(Defined recursively)

- **Base case:** $q \in \varepsilon\text{-REACHABLE}(q)$

- **Inductive case:**

A state is in the reachable set if …

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

… there is an empty transition to it from another state in the reachable set

With empty transitions

Last Time: NFA Extended Transition Function

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$

Domain:

- Beginning state $q \in Q$
- Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$

Range:

- Ending set of states

(Defined recursively, on length of input string)

- **Base case:** $\hat{\delta}(q, \epsilon) = \cancel{\{q\}}$ $\varepsilon\text{-REACHABLE}(q)$
- **Recursive case:** $\hat{\delta}(q, w) =$ ~~$\bigcup_{i=1}^{k} \delta(q_i, w_n)$~~ $\varepsilon\text{-REACHABLE}(\bigcup_{i=1}^{k} \delta(q_i, w_n))$

“For all current states, take single step, then follow all empty transitions”

where: $\hat{\delta}(q, w_1 \cdots w_{n-1}) = \{q_1, \ldots, q_k\}$

Concatenation: $A \circ B = \{xy|\ x \in A \text{ and } y \in B\}$

Last Time: Concatenation is Closed?

THEOREM

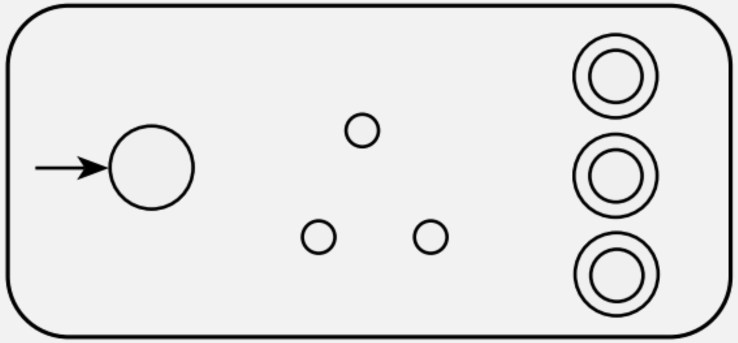
The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

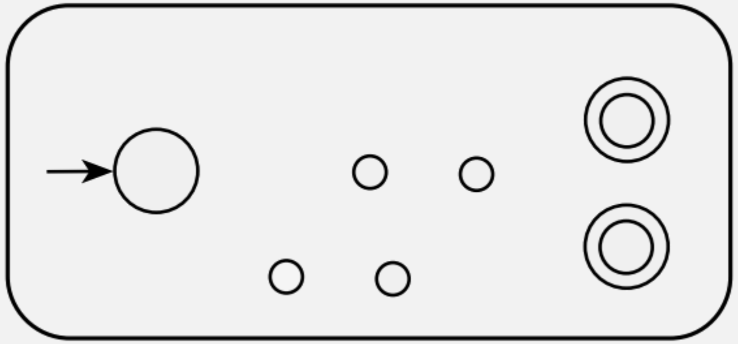
Proof: Construct ~~a new machine~~ an NFA!

Concatentation

DFA N_1



DFA N_2



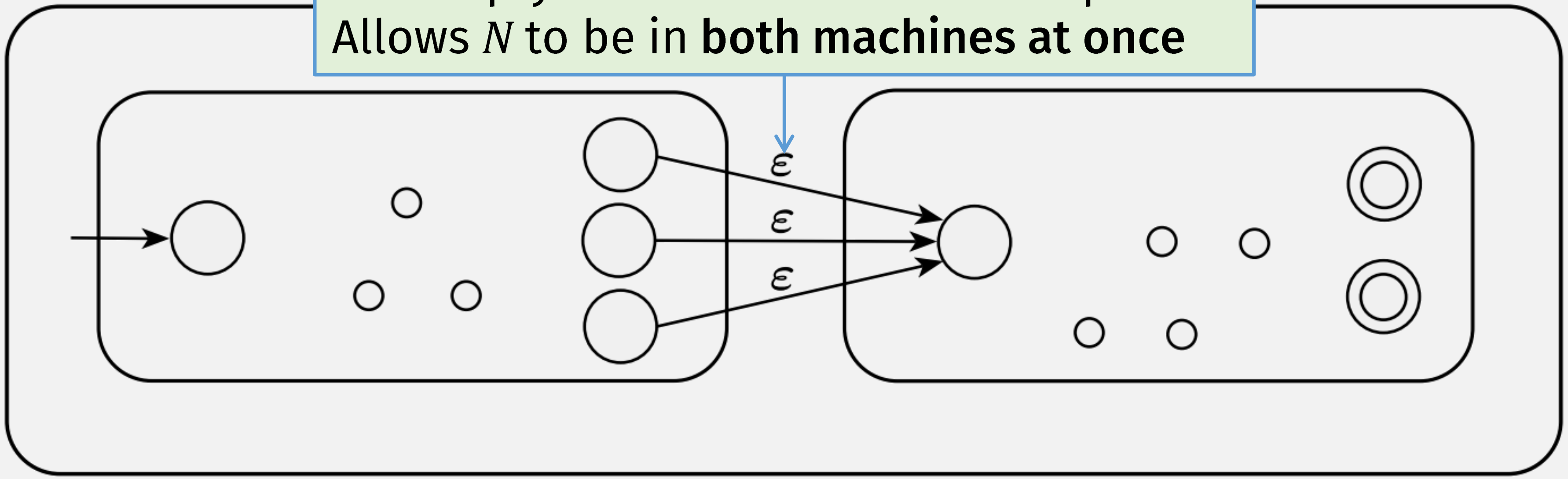
Let N_1 recognize A_1, and N_2 recognize A_2.

Want: Construction of N to recognize $A_1 \circ A_2$

N is an NFA! It simultaneously:
- Keeps checking 1st part with N_1
  and
- Moves to N_2 to check 2nd part

NFA N

ε = "empty transition" = reads no input
Allows N to be in **both machines at once**



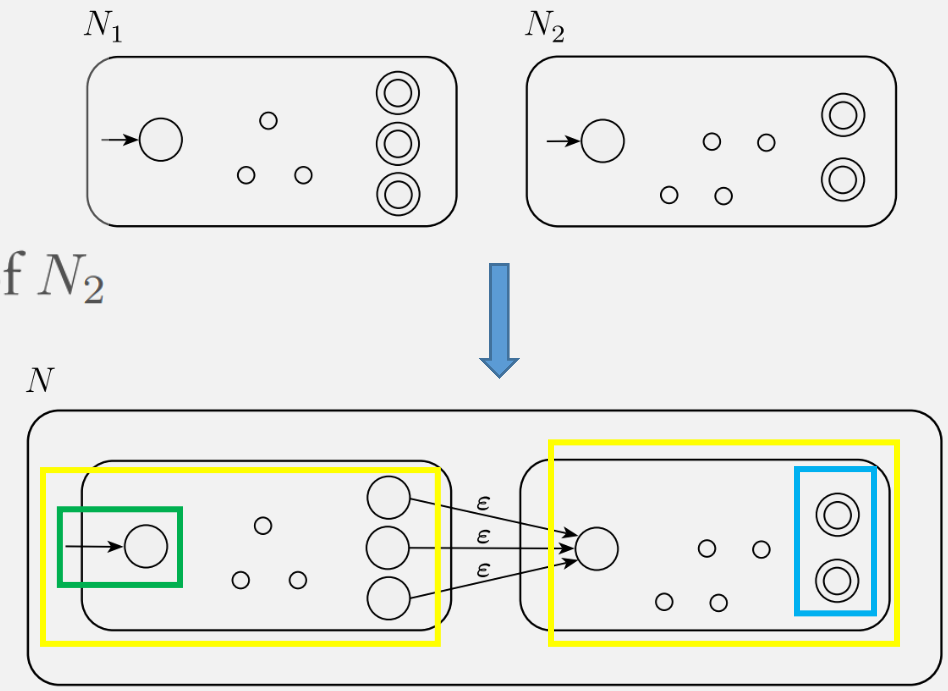
Concatenation is Closed for Regular Langs

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, \boxed{q_1}, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (\boxed{Q}, \Sigma, \delta, \boxed{q_1}, \boxed{F_2})$ to recognize $A_1 \circ A_2$

1. $\boxed{Q} = Q_1 \cup Q_2$
2. The state $\boxed{q_1}$ is the same as the start state of N_1
3. The accept states $\boxed{F_2}$ are the same as the accept states of N_2



Concatenation is Closed for Regular Langs

Wait, is this true?

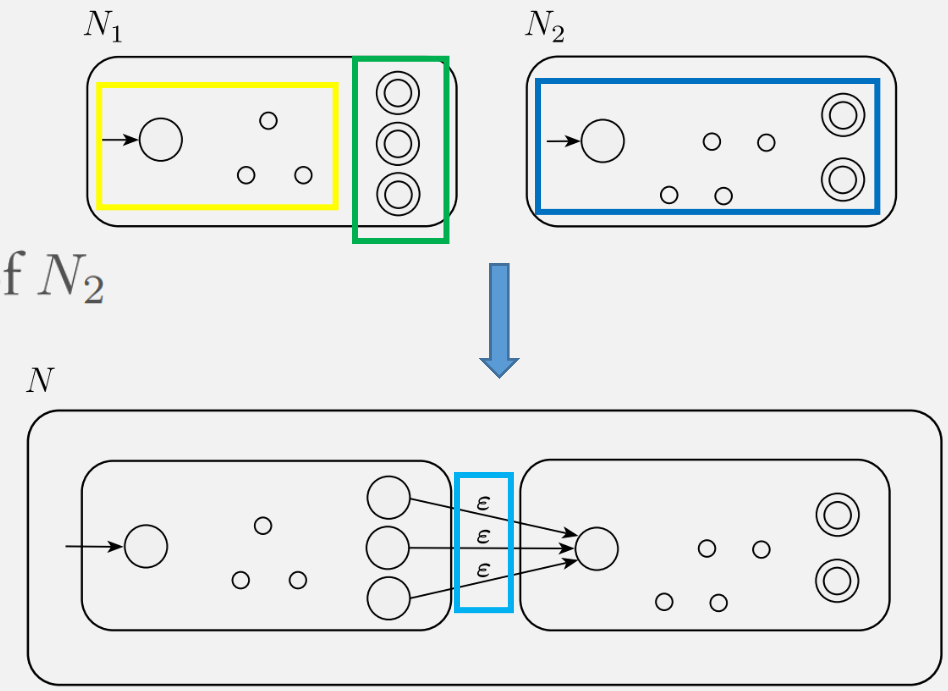
PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state q_1 is the same as the start state of N_1
3. The accept states F_2 are the same as the accept states of N_2
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(\overset{?}{q}, a) & \boxed{q \in Q_1 \text{ and } q \notin F_1} \\ \delta_1(\overset{?}{q}, a) & \boxed{q \in F_1 \text{ and } a \neq \varepsilon} \\ \delta_1(\overset{?}{q}, a) \cup \{q_2\} & q \in F_1 \text{ and } \boxed{a = \varepsilon} \\ \delta_2(\overset{?}{q}, a) & \boxed{q \in Q_2.} \end{cases}$$



??? ■

Flashback: A DFA’s Language

- For DFA $M = (Q, \Sigma, \delta, q_0, F)$
- M ***accepts*** w if $\hat{\delta}(q_0, w) \in F$
- M ***recognizes language*** A if $A = \{w |\ M \text{ accepts } w\}$

- A DFA’s language is a **regular language**

An NFA's Language

- For NFA $N = (Q, \Sigma, \delta, q_0, F)$

intersection

accept states

- N ***accepts*** w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$

not empty

  - i.e., final states have at least one accept state

- Language of N = $L(N)$ = $\left\{ w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \right\}$

Q: An NFA's language is a __regular?__ language

Concatenation Closed for Reg Langs?

- Concatenation of DFAs produces an NFA
- But a language is only regular if a DFA recognizes it
- So to finish the proof that concatenation is closed …
  … we must prove that NFAs *also* recognize regular languages.

Specifically, we must prove:

NFAs ⇔ regular languages

How to Prove a Statement: $X \Leftrightarrow Y$

$X \Leftrightarrow Y$ = "X if and only if Y" = X iff Y = X <=> Y

Proof at minimum has 2 required parts:

1. $\Rightarrow$ if X, then Y
    - "**forward**" direction
    - assume X, then use it to prove Y
2. $\Leftarrow$ if Y, then X
    - "**reverse**" direction
    - assume Y, then use it to prove X

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular if and only if some NFA N recognizes L.

Proof:

⇒ If **L is regular**, then **some NFA N recognizes it.**

(Easier)

- We know: **if L is regular, then a DFA exists that recognizes it.**
- So to prove this part: **Convert that DFA to an equivalent NFA!** (see HW 2)

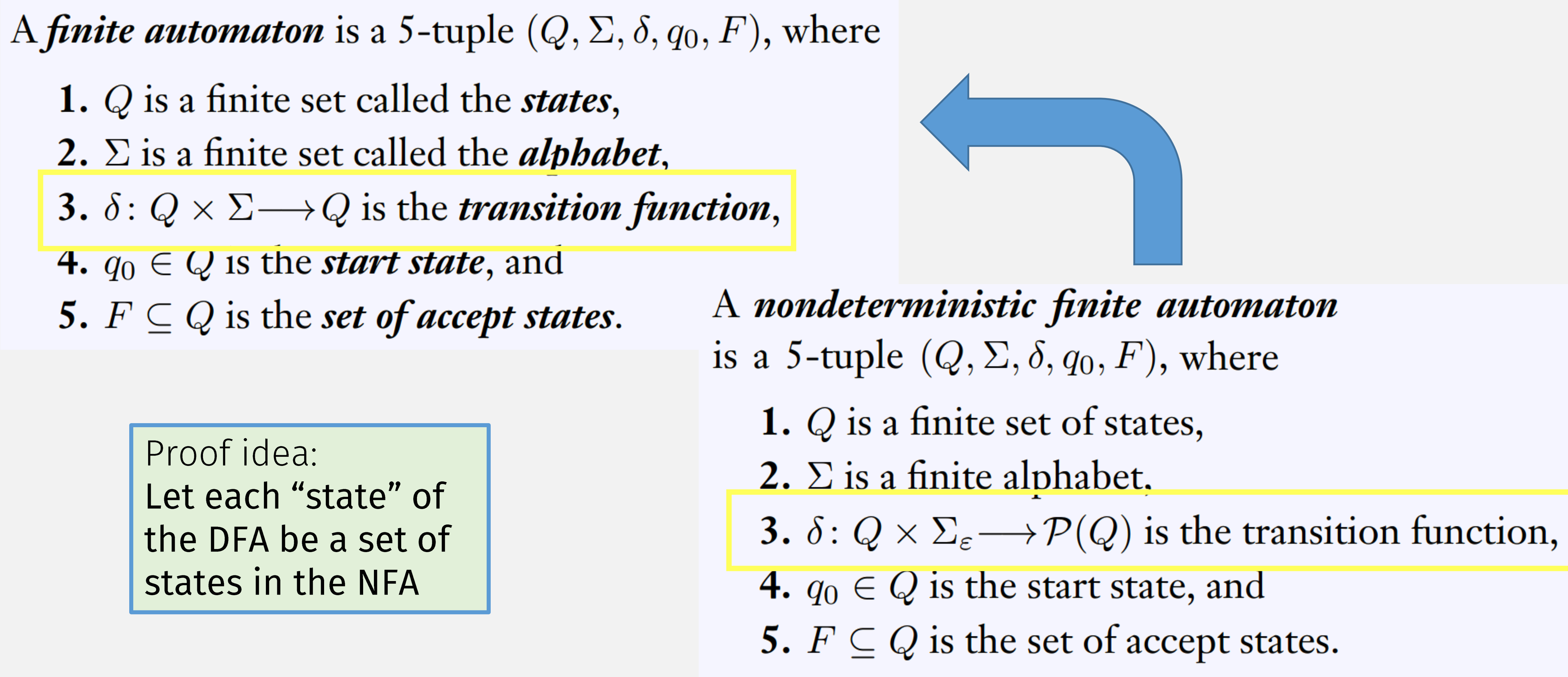
⇐ If **an NFA N recognizes L**, then **L is regular.**

(Harder)

- We know: **for L to be regular, there must be a DFA recognizing it**
- Proof Idea for this part: **Convert given NFA N to an equivalent DFA**

"equivalent" = "recognizes the same language"

How to convert **NFA→DFA**?



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A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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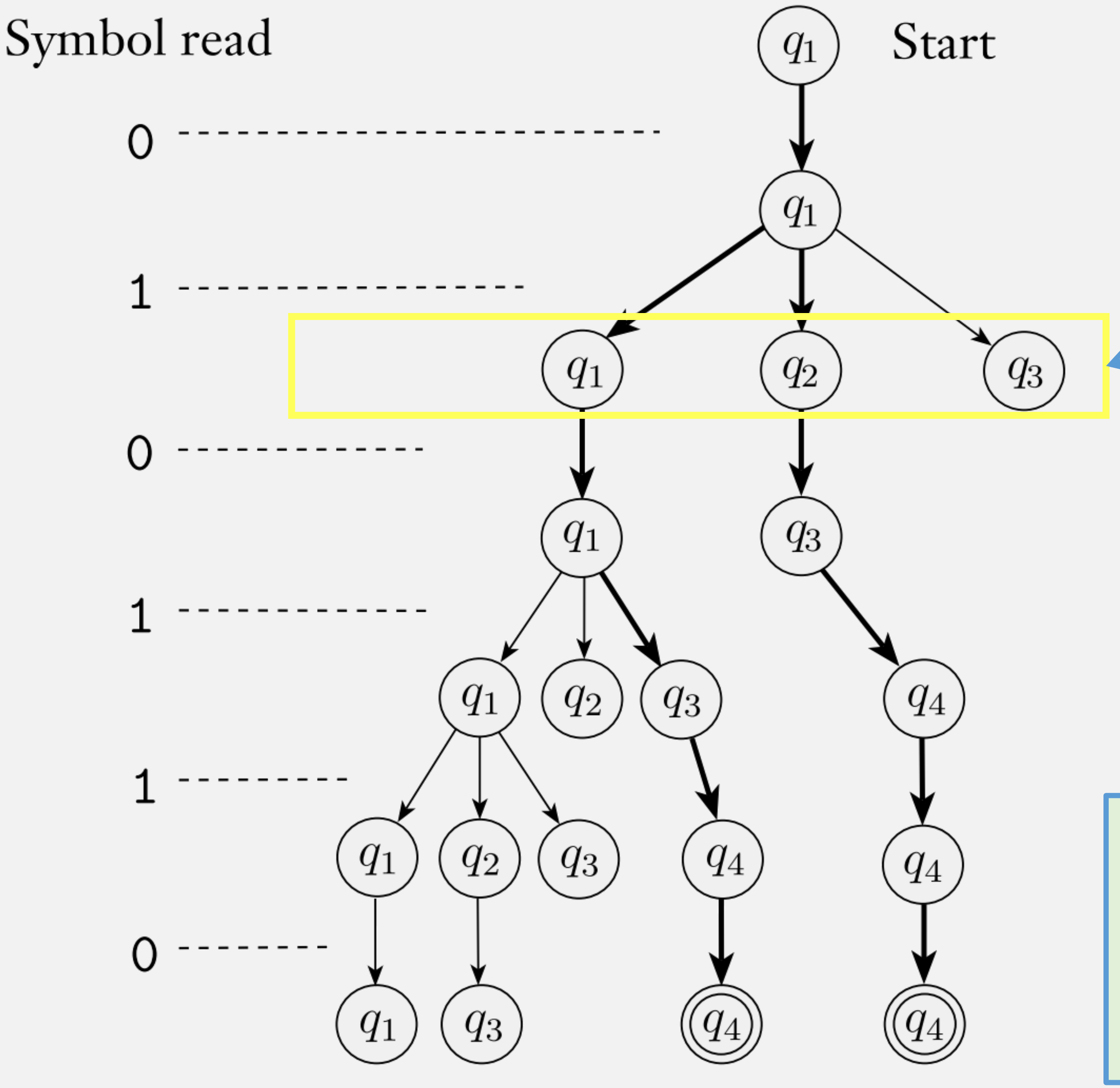
3. $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Proof idea:
Let each "state" of the DFA be a set of states in the NFA



In a DFA, all these states at each step of NFA computation must be only **one** state

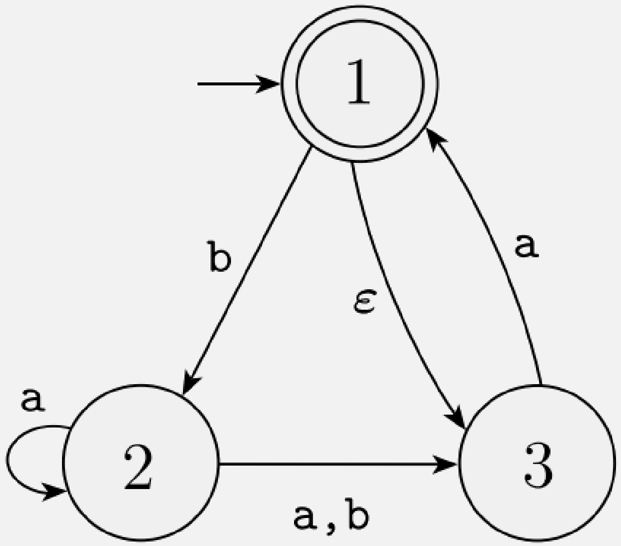
So encode:
a <u>set of NFA states</u>
as <u>one DFA state</u>

This is similar to the proof strategy from **"Closure of union"** where:
a state = a pair of states

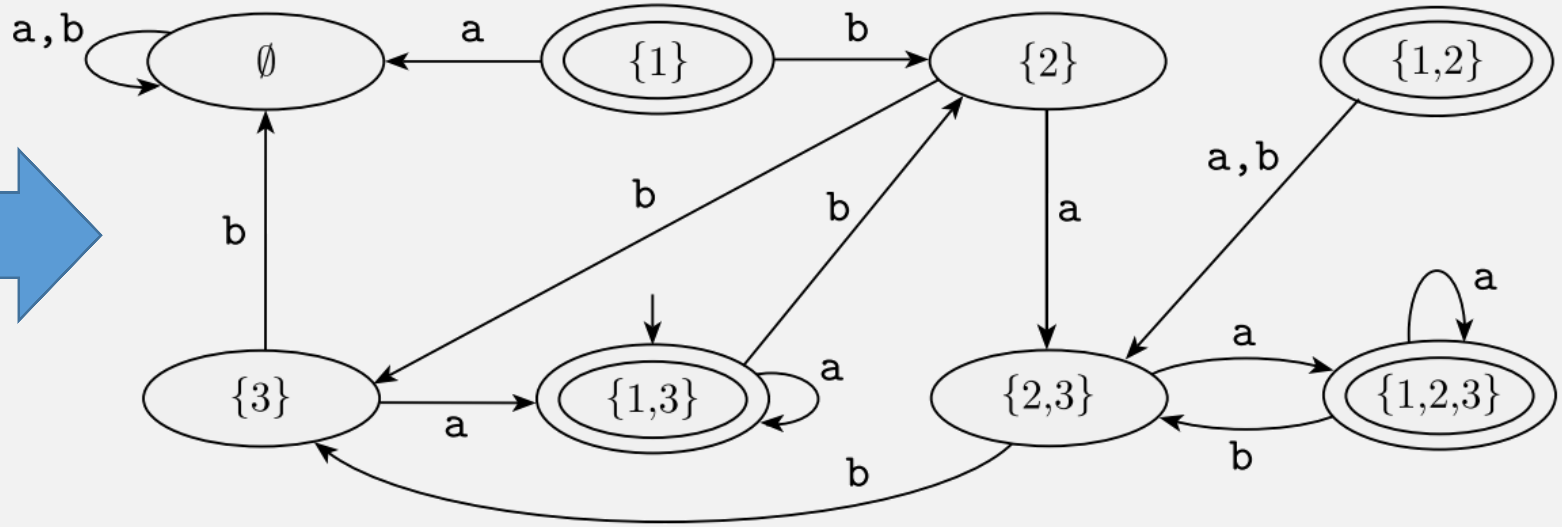
Convert **NFA→DFA**, Formally

- Let **NFA** *N* = $(Q, \Sigma, \delta, q_0, F)$
- An equivalent DFA *M* has states Q' = $\mathcal{P}(Q)$ (power set of Q)

Example:



The NFA N_4



A DFA D that is equivalent to the NFA N_4

No empty transitions

NFA→DFA

Have: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Want: DFA $M = (Q', \Sigma, \delta', {q_0}', F')$

1. $Q' = \mathcal{P}(Q)$

A state for M is a set of states in N

2. For $R \in Q'$ and $a \in \Sigma$,

R = a state in M = a set of states in N

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

Next state for DFA state R =
next states of each NFA state r in R

3. ${q_0}' = \{q_0\}$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

Flashback: Adding Empty Transitions

- Define the set $\varepsilon\text{-}\mathrm{REACHABLE}(q)$
  - … to be all states reachable from q via zero or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-}\mathrm{REACHABLE}(q)$

- Inductive case:

A state is in the reachable set if …

$$\varepsilon\text{-}\mathrm{REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-}\mathrm{REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

… there is an empty transition to it from another state in the reachable set

With empty transitions

NFA→DFA

Have: NFA $N = (Q, \Sigma, \delta, q_0, F)$

Want: DFA $M = (Q', \Sigma, \delta', {q_0}', F')$

Almost the same, except ...

1. $Q' = \mathcal{P}(Q)$

2. For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \cancel{\delta(r, a)} \ \varepsilon\text{-REACHABLE}(\delta(r, a))$$

3. ${q_0}' = \cancel{\{q_0\}} \ \varepsilon\text{-REACHABLE}(q_0)$

4. $F' = \{R \in Q' \mid R$ contains an accept state of $N\}$

Proving NFAs Recognize Regular Langs

Theorem:

A language L is regular **if and only if** some NFA N recognizes L.

Proof:

⇒ If L is regular, then some NFA N recognizes it.

- We know: If L is **regular**, then a **DFA** recognizes it.
- We show: How to convert a DFA to an equivalent NFA (proved in hw2)

⇐ If an NFA N recognizes L, then L is regular.

- We know: For L to be **regular**, there must be a **DFA** recognizing it
- We show: How to convert NFA N to an equivalent DFA …
- … using the NFA→DFA algorithm we just defined!

■

Concatenation is Closed for Regular Langs

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

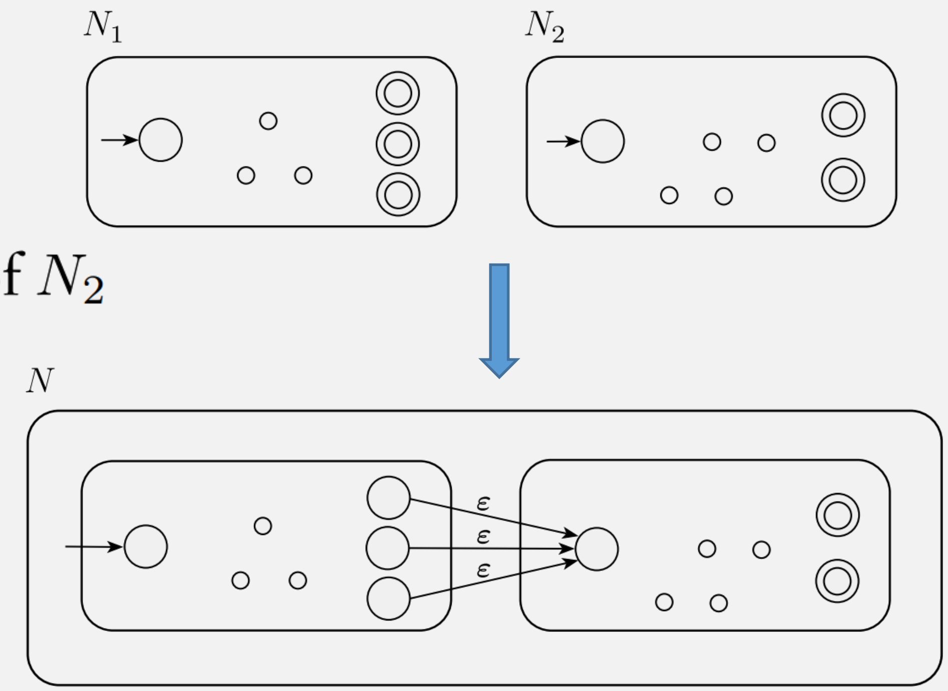
1. $Q = Q_1 \cup Q_2$

2. The state q_1 is the same as the start state of N_1

3. The accept states F_2 are the same as the accept states of N_2

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \boldsymbol{\varepsilon} \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \boldsymbol{\varepsilon} \\ \delta_2(q,a) & q \in Q_2. \end{cases}$$



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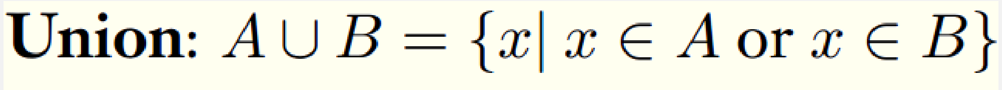
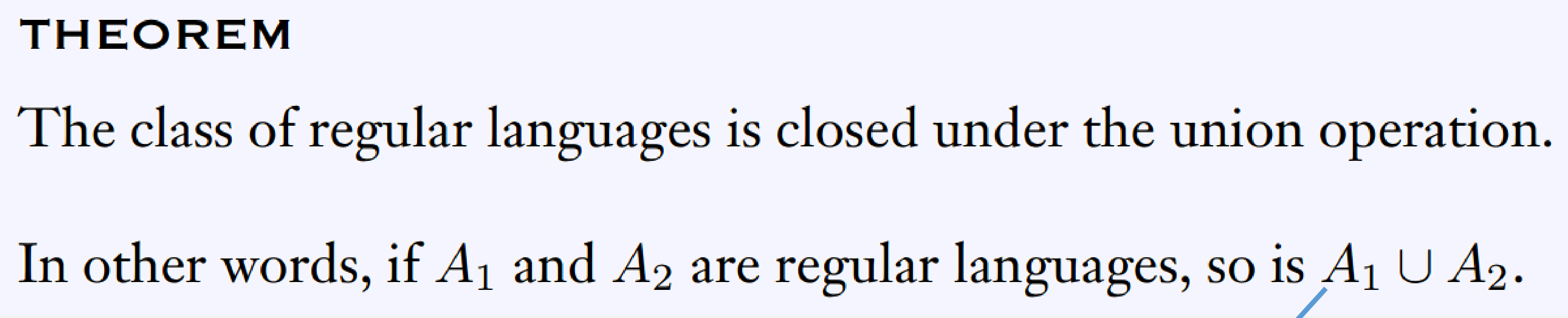


Union: $A \cup B = \{x |\ x \in A \text{ or } x \in B\}$

Flashback: Union is Closed For Regular Langs



THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Proof:

- How do we prove that a language is regular?
  - Create a DFA or NFA recognizing it!
- Combine the machines recognizing A_1 and A_2
  - Should we create a DFA or NFA?

Flashback: Union is Closed For Regular Langs

Proof

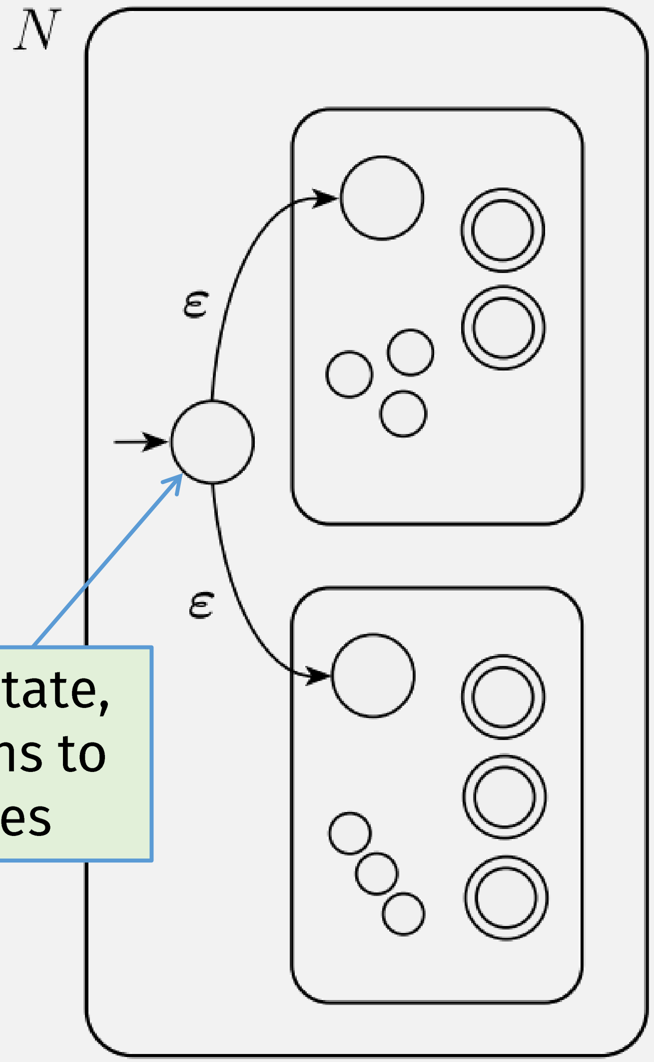
- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: **a new machine** $M = (Q, \Sigma, \delta, q_0, F)$ **using** M_1 **and** M_2
- states of M: $Q = \{(r_1, r_2) | \, r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2
- M transition fn: $\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big)$
- M start state: (q_1, q_2)
- M accept states: $F = \{(r_1, r_2) | \, r_1 \in F_1 \text{ or } r_2 \in F_2\}$

State in M =
M_1 state +
M_2 state

M step =
a step in M_1 + a step in M_2

Accept if <u>either</u> M_1 or M_2 accept

Union is Closed for Regular Languages



Add new start state, and ε-transitions to old start states

Alternate Proof, with NFAs

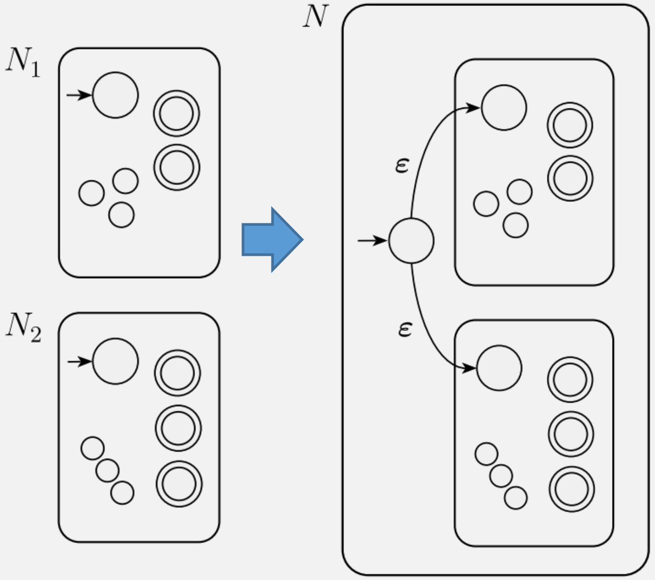
Union is Closed for Regular Languages

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.



Alternate Proof, with NFAs

Union is Closed for Regular Languages

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
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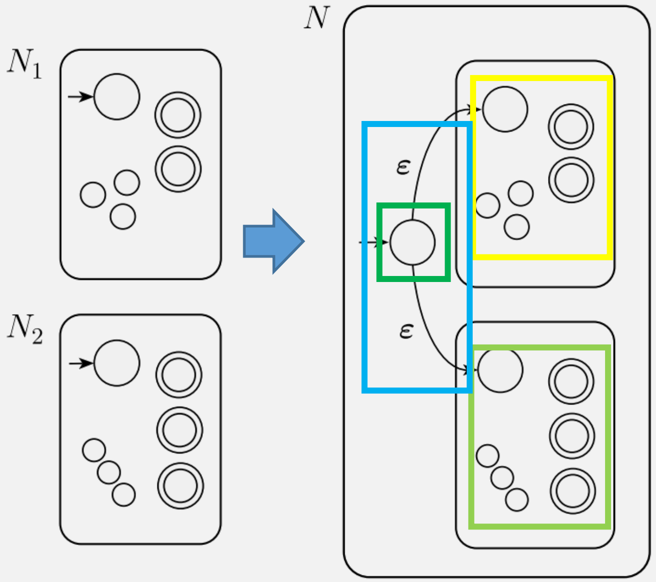
Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} \delta_1(?, a) & \boxed{q \in Q_1} \\ \delta_2(?, a) & \boxed{q \in Q_2} \\ \{q_1 ? q_2\} & \boxed{q = q_0 \text{ and } a = \varepsilon} \\ \emptyset \quad ? & \boxed{q = q_0 \text{ and } a \neq \varepsilon} \end{cases}$$



List of Closed Ops for Reg Langs (so far)

- ☑ Union
- ☑ Concatentation
- Kleene Star (repetition)

Star: $A^* = \{x_1 x_2 \ldots x_k | \; k \geq 0 \text{ and each } x_i \in A\}$

Kleene Star Example

Let the alphabet Σ be the standard 26 letters $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$.

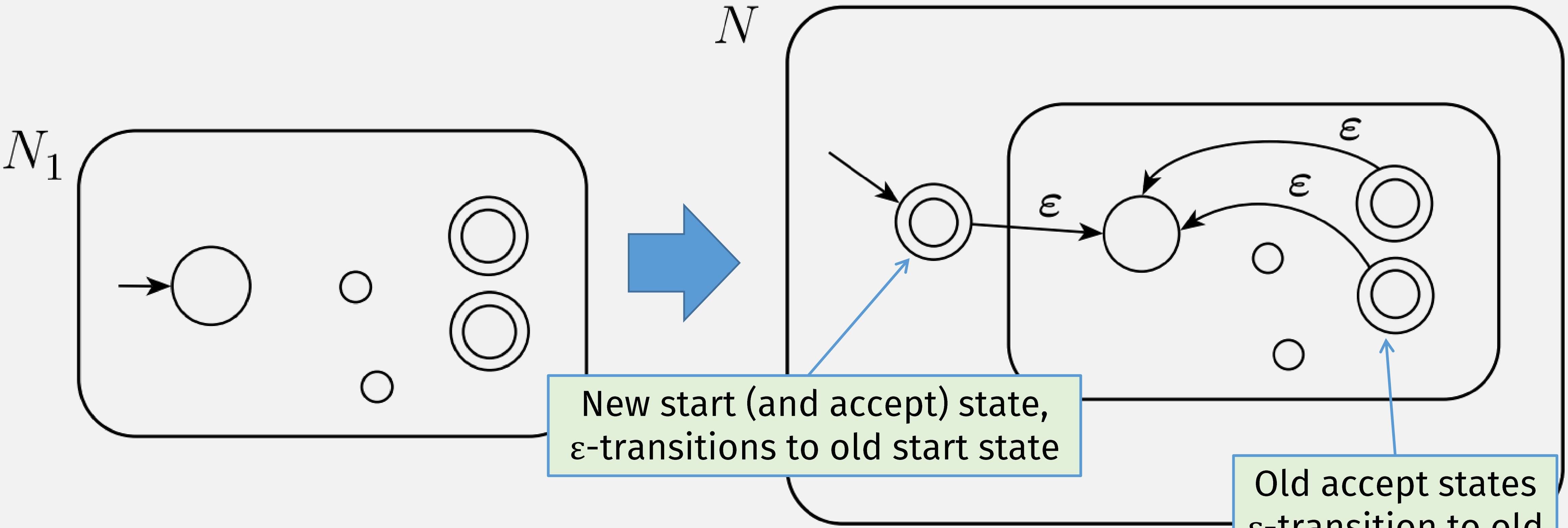
If $A = \{\texttt{good}, \texttt{bad}\}$ and $B = \{\texttt{boy}, \texttt{girl}\}$, then

$$A^* = \{\varepsilon, \text{good, bad, goodgood, goodbad, badgood, badbad, goodgoodgood, goodgoodbad, goodbadgood, goodbadbad}, \ldots\}$$

Note: **repeat zero or more times**

(this is an infinite language!)

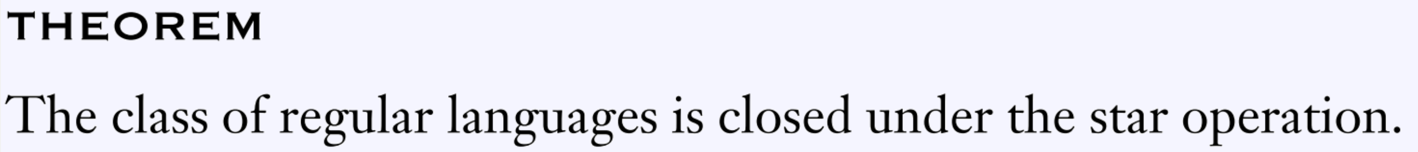
Kleene Star



New start (and accept) state, ε-transitions to old start state

Old accept states ε-transition to old start state

Kleene Star is Closed for Regular Langs



THEOREM

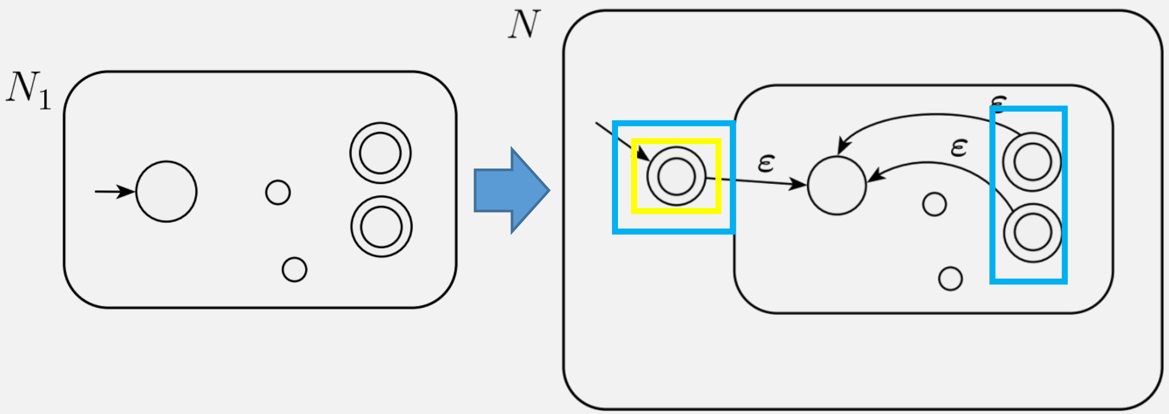
The class of regular languages is closed under the star operation.

Kleene Star is Closed for Regular Langs

PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.

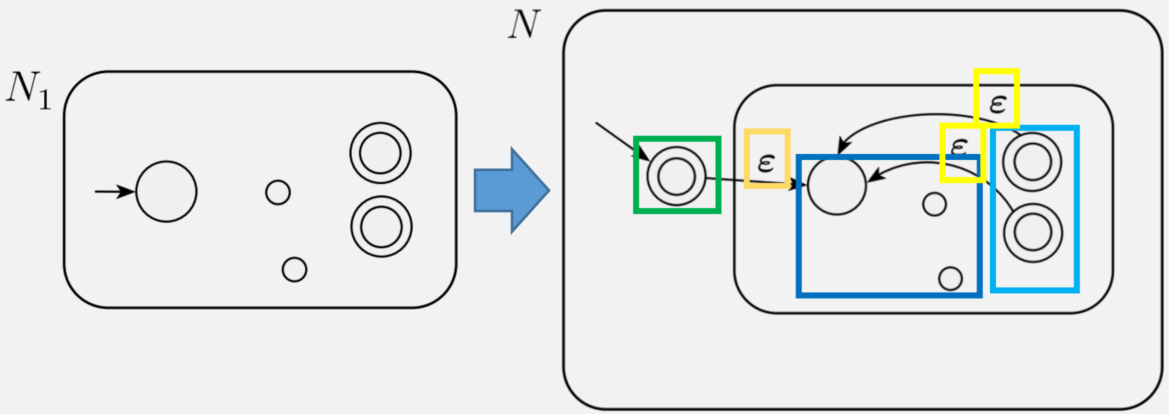
1. $Q = \boxed{\{q_0\}} \cup Q_1$
2. The state $\boxed{q_0}$ is the new start state.
3. $F = \boxed{\{q_0\} \cup F_1}$

Kleene star of a language must accept the empty string!



Kleene Star is Closed for Regular Langs

PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.



1. $Q = \{q_0\} \cup Q_1$

2. The state q_0 is the new start state.

3. $F = \{q_0\} \cup F_1$

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a)\,? & \boxed{q \in Q_1 \text{ and } q \notin F_1} \\ \delta_1(q,a)\,? & \boxed{q \in F_1 \text{ and } a \neq \varepsilon} \\ \delta_1(q,a)\,? \cup \{q_1\} & q \in F_1 \text{ and } \boxed{a = \varepsilon} \\ \{q_1\} \quad ? & q = q_0 \text{ and } \boxed{a = \varepsilon} \\ \emptyset \quad ? & \boxed{q = q_0 \text{ and } a \neq \varepsilon.} \end{cases}$$

Many More Closed Operations on Regular Languages!

- Complement
- Intersection
- Difference
- Reversal
- Homomorphism
- (See HW2)

Why do we care about these ops?

- Union
- Concat
- Kleene star

- The are sufficient to represent <u>all regular languages!</u>

- I.e., they define **regular expressions**

Check-in Quiz 9/27

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