Examples with the Pumping Lemma

Wed Feb 24, 2021
Logistics

• HW3 solutions posted (soon)

• HW4 due Sunday 2/28 11:59pm EST

• Questions?
**Last time:** The Pumping Lemma says:

**Pumping Lemma** If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^i z \in A$, 
2. $|y| > 0$, and 
3. $|xy| \leq p$.

...these strings must be divisible into three pieces (call them $x$, $y$, and $z$) ...

...where repeating the middle piece $y$ results in a “pumped” string is also in the language

Also, repeating part:  
- can’t be empty string  
- must be in the first $p$ characters

**tl;dr:** Long enough strings means repeated states
Last time: Equivalence of Contrapositive

• “If X then Y” is equivalent to ... ?

  • “If Y then X” (converse)
    • No!

  • “If not X then not Y” (inverse)
    • No!

✓“If not Y then not X” (contrapositive)
  • Yes!
    • Proof by contradiction uses this equivalence
The Pumping Lemma is an If-Then Stmt

**Pumping lemma** If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Just need one counterexample!

Contrapositive: If (any of) these are not true ...
Pumping Lemma: Non-Regularity Example

Let $B$ be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that $B$ is not regular. The proof is by contradiction.

**Pumping lemma** If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$. 
Theorem
The language \( B = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof.
This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.
Theorem
The language $B = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof.
This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. State the kind of proof: The proof is by contradiction.

2. State assumptions: Assume that $B$ is a regular language.
   Then it must satisfy the pumping lemma where $p$ is the pumping length.
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1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that $B$ is a regular language.
   Then it must satisfy the pumping lemma where $p$ is the pumping length.
3. Present counterexample: Choose $s$ to be the string $0^p1^p$.

5. The contradiction step typically requires detailed case analysis of scenarios.
   There are three possible cases:
   5.1 $y$ is all 0s: Pumped strings, e.g., $xyyz$, are not in $B$ because they have more 0s than 1s, breaking condition 1 of the pumping lemma. So we have a contradiction.
   5.2 $y$ is all 1s: Same as above.
   5.3 $y$ has both 0s and 1s: Pumped strings preserve equal counts, but is out of order and therefore not in $B$, breaking condition 1.

6. Conclusion: Since all cases result in contradiction, $B$ must not be regular.
Theorem

The language $B = \{0^n1^n \mid n \geq 0\}$ is not regular.

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3. Present counterexample: Choose $s$ to be the string $0^p1^p$.

4. Show contradiction of assumption: Because $s \in B$ and has length $> p$, the
   pumping lemma guarantees that $s$ can be split into three pieces $s = xyz$ where
   $xy^iz \in B$ for $i \geq 0$. But we will show this is impossible ...
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1. **State the kind of proof:** The proof is by contradiction.

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   5.1. **\( y \) is all 0s:** Pumped strings, e.g., \( xy^yz \), are not in \( B \) because they have more 0s than 1s, breaking condition 1 of the pumping lemma. So we have a contradiction.

   5.2. **\( y \) is all 1s:** Same as above.

   5.3. **\( y \) has both 0s and 1s:** Pumped strings preserve equal counts, but is out of order and therefore not in \( B \), breaking condition 1.

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   $xy^iz \in B$ for $i \geq 0$. But we will show this is impossible ...
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Theorem

*The language $B = \{0^n1^n \mid n \geq 0\}$ is not regular.*

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This proof is annotated with *commentary in blue.* (Commentary not needed for hw proofs.)

1. **State the kind of proof:** The proof is by contradiction.

2. **State assumptions:** Assume that $B$ is a regular language.

   Then it must satisfy the pumping lemma where $p$ is the pumping length.

3. **Present counterexample:** Choose $s$ to be the string $0^p1^p$.

4. **Show contradiction of assumption:** Because $s \in B$ and has length $> p$, the pumping lemma guarantees that $s$ can be split into three pieces $s = xyz$ where $xy^iz \in B$ for $i \geq 0$. But we will show this is impossible ...

5. **The contradiction step typically requires detailed case analysis of scenarios.** There are three possible cases:

   5.1 *y* is all 0s: Pumped strings, e.g., $xyyz$, are not in $B$ because they have more 0s than 1s, breaking condition 1 of the pumping lemma. So we have a contradiction.

   5.2 *y* is all 1s: Same as above.

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   5.3 \( y \) has both 0s and 1s: Pumped strings preserve equal counts, but is out of order and therefore not in \( B \), breaking condition 1.

6. Alternate Proof: Last 2 cases not needed; see pumping lemma, condition 3.
Possible Split: $y = \text{all 0s}$

- **Assumption**: $0^n1^n$ is a regular language (must satisfy pumping lemma)
- **Counterexample** = $0^p1^p$

If $xyz$ chosen so $y$ contains
  - all 0s

```
  00 ... 011 ... 1
```

- **Pumping $y$**: produces a string with more 0s than 1s
  - This string is **not** in the language $0^n1^n$
  - This means that $0^n1^n$ does **not** satisfy the pumping lemma
  - Which means that that $0^n1^n$ is a **not** regular lang
  - This is a **contradiction** of the assumption!

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**Pumping lemma** If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$. 

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But pumping lemma requires **only one** pumpable splitting

So we must show that every splitting produces a contradiction
**Possible Split:** $y = \text{all 1s}$

- **Assumption:** $0^n1^n$ is a regular language (must satisfy pumping lemma)
- **Counterexample** = $0^p1^p$
- If $xyz$ chosen so $y$ contains
  - all 1s

```
  \[
  \begin{array}{c}
  p \ 0s \\
  \hline \\
  00 \ldots 011 \ldots 1 \\
  \hline \\
  x \quad y \quad z
  \end{array}
  \]
```

- Is this string pumpable?
  - No!
  - By the same reasoning as in the previous slide
Possible Split: $y = 0s$ and $1s$

- **Assumption:** $0^n1^n$ is a regular language (must satisfy pumping lemma)
- **Counterexample** = $0^p1^p$
  
  $00 \ldots 011 \ldots 1$

- If $xyz$ chosen so $y$ contains
  - both $0s$ and $1s$

- Is this string pumpable?
  - No!
  - Pumped string will have equal $0s$ and $1s$
  - But they will be in the wrong order: so there is still a **contradiction**!
**Last time:** The Pumping Lemma says:

**Pumping lemma**  If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

**Also,** repeating party:
- can’t be empty string
- must be in the first $p$ characters

\[00 \ldots 011 \ldots 1\]

$y$ must be in here!
Pumping Lemma: How to use Condition 3

Let $F = \{ww | w \in \{0,1\}^*\}$. We show that $F$ is nonregular.
Using Condition 3 of the Pumping Lemma

Theorem

The language \( F = \{ww \mid w \in \{0,1\}^*\} \) is not regular.

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. **State the kind of proof:** The proof is by contradiction.

2. **State assumptions:** Assume that \( F \) is regular. Then it must satisfy the pumping lemma where \( p \) is the pumping length.

3. **Present counterexample:** Choose \( s \) to be the string \( 0^p10^p1 \).

4. **Show contradiction of assumption:** Because \( s \in F \) and has length \( > p \), the pumping lemma guarantees that \( s \) can be split into three pieces \( s = xyz \) where \( xy^i z \in F \) for \( i \geq 0 \). But we will show this is impossible...

5. **This time there is only one possible case, but we must explain why.** According to condition 3 of the pumping lemma \( |xy| \leq p \). So \( y \) is all 0s. But then \( xyyz \not\in F \), breaking condition 1 of the pumping lemma. So we have a contradiction.

6. **Conclusion:** Since all cases result in contradiction, \( F \) must not be regular.
Theorem

The language $F = \{ww | w \in \{0,1\}^*\}$ is not regular.

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. **State the kind of proof:** The proof is by contradiction.

2. **State assumptions:** Assume that $F$ is regular. Then it must satisfy the pumping lemma where $p$ is the pumping length.

3. **Present counterexample:** Choose $s$ to be the string $0^p10^p1$.

4. **Show contradiction of assumption:** Because $s \in F$ and has length $> p$, the pumping lemma guarantees that $s$ can be split into three pieces $s = xyz$ where $xyi \in F$ for $i \geq 0$. But we will show this is impossible ...

5. **This time there is only one possible case, but we must explain why.** According to condition 3 of the pumping lemma $|xy| \leq p$. So $y$ is all 0s. But then $xyyz \notin F$, breaking condition 1 of the pumping lemma. So we have a contradiction.

6. **Conclusion:** Since all cases result in contradiction, $F$ must not be regular.
Using Condition 3 of the Pumping Lemma

Theorem

The language $F = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

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1. State the kind of proof: The proof is by contradiction.
2. State assumptions: Assume that $F$ is regular. Then it must satisfy the pumping lemma where $p$ is the pumping length.
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Using Condition 3 of the Pumping Lemma

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3. **Present counterexample:** Choose $s$ to be the string $0^p10^p1$.
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The language \( F = \{ww \mid w \in \{0, 1\}^*\} \) is not regular.

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5. This time there is only one possible case, but we must explain why. According to condition 3 of the pumping lemma \( |xy| \leq p \). So \( y \) is all 0s. But then \( xyyz \notin F \), breaking condition 1 of the pumping lemma. So we have a contradiction.
Using Condition 3 of the Pumping Lemma

Theorem
The language $F = \{ww \mid w \in \{0, 1\}^*\}$ is not regular.

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5. This time there is only one possible case, but we must explain why. According to condition 3 of the pumping lemma $|xy| \leq p$. So $y$ is all 0s. But then $xyyz \not\in F$, breaking condition 1 of the pumping lemma. So we have a contradiction.
6. Conclusion: Since all cases result in contradiction, $F$ must not be regular.
Pumping Lemma: Pumping Down

use the pumping lemma to show that \( E = \{0^i1^j | i > j \} \) is not regular.
Theorem

The language $E = \{0^i1^j \mid i > j\}$ is not regular.

Proof.

This proof is annotated with commentary in blue. (Commentary not needed for hw proofs.)

1. **State the kind of proof:** The proof is by contradiction.
Pumping Down

Theorem

The language \( E = \{0^i1^j \mid i > j\} \) is not regular.

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\(\)
Theorem

The language $E = \{0^i1^j \mid i > j\}$ is not regular.

Proof.

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1. State the kind of proof: The proof is by contradiction.

2. State assumptions: Assume that $E$ is regular. Then it must satisfy the pumping lemma where $p$ is the pumping length.

3. Present counterexample: Choose $s$ to be the string $0^{p+1}1^p$.

4. Show contradiction of assumption: Because $s \in E$ and has length $> p$, the pumping lemma guarantees that $s$ can be split into three pieces $s = xyz$ where $xy^iz \in E$ for $i \geq 0$. But we will show this is impossible ...

5. Again, one possible case. According to condition 3 of the pumping lemma $|xy| \leq p$. So $y$ is all 0s. But then $xz/ \notin E$ ($i = 0$), breaking condition 1 of the pumping lemma. So we have a contradiction.

6. Conclusion: Since all cases result in contradiction, $E$ must not be regular.
Theorem

The language $E = \{0^i1^j \mid i > j\}$ is not regular.

Proof.

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1. State the kind of proof: The proof is by contradiction.

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5. **Again, one possible case.** According to condition 3 of the pumping lemma $|xy| \leq p$. So $y$ is all 0s. But then $xz \notin E$ ($i = 0$), breaking condition 1 of the pumping lemma. So we have a contradiction.
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Check-in Quiz 2/24

On gradescope