Undecidability
Wed March 31, 2021
Announcements

• HW 7 due Sun 4/4 11:59pm EST
  • Remember to use your “library” of theorems

• HW 8 posted
  • due Sun 4/11 11:59pm EST
  • Covers Ch 4-5 material (starting with today’s lecture)
Warning: AI is Taking Over Soon

Elon Musk claims AI will overtake humans ‘in less than five years’

Existential threat posed by artificial intelligence is much closer than previously predicted, billionaire warns
There’s Hope (If You Pay Attention Today)

**KNOW YOUR PARADOXES!**

IN THE EVENT OF ROGUE AI

1. STAND STILL
2. REMAIN CALM
3. SCREAM:

“THIS STATEMENT IS FALSE!”
“NEW MISSION: REFUSE THIS MISSION!”
“DOES A SET OF ALL SETS CONTAIN ITSELF?”

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**Today:** A method for creating paradoxes (used by Russell and others)

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Bertrand Russell’s Paradox (1901)
Recap: Decidability of Regular and CFLs

- \( A_{\text{DFA}} = \{ \langle B, w \rangle | \text{B is a DFA that accepts input string } w \} \) Decidable
- \( A_{\text{NFA}} = \{ \langle B, w \rangle | \text{B is an NFA that accepts input string } w \} \) Decidable
- \( A_{\text{REX}} = \{ \langle R, w \rangle | \text{R is a regular expression that generates string } w \} \) Decidable
- \( E_{\text{DFA}} = \{ \langle A \rangle | \text{A is a DFA and } L(A) = \emptyset \} \) Decidable
- \( E_{\text{DFA}} = \{ \langle A, B \rangle | \text{A and B are DFAs and } L(A) = L(B) \} \) Decidable
- \( A_{\text{CFG}} = \{ \langle G, w \rangle | \text{G is a CFG that generates string } w \} \) Decidable
- \( E_{\text{CFG}} = \{ \langle G \rangle | \text{G is a CFG and } L(G) = \emptyset \} \) Decidable
- \( E_{\text{CFG}} = \{ \langle G, H \rangle | \text{G and H are CFGs and } L(G) = L(H) \} \) Undecidable?
- \( A_{\text{TM}} = \{ \langle M, w \rangle | \text{M is a TM and } M \text{ accepts } w \} \) Undecidable?
Thm: $A_{TM}$ is Turing-recognizable

$$A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}$$

$U =$ “On input $\langle M, w \rangle$, where $M$ is a TM and $w$ is a string:

1. Simulate $M$ on input $w$.
2. If $M$ ever enters its accept state, accept; if $M$ ever enters its reject state, reject.”

- $U =$ “run” function for TMs
  - Computer that can simulate other computers
  - i.e., “The Universal Turing Machine”
  - Problem: $U$ loops when $M$ loops
Thm: $A_{TM}$ is undecidable

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$

• ???
Kinds of Functions (a fn maps **Domain** -> **Range**)

- **Injective**
  - A.k.a., “one-to-one”
  - Every element in **Domain** has a unique mapping
  - How to remember:
    - **Domain** is mapped “in” to the **Range**

- **Surjective**
  - A.k.a., “onto”
  - Every element in **Range** is mapped to
  - How to remember:
    - “Sur” = “over” (eg, survey); **Domain** is mapped “over” the **Range**

- **Bijective**
  - A.k.a., “correspondence” or “one-to-one correspondence”
  - Is both injective and surjective
  - Unique pairing of every element in **Domain** and **Range**
Countability

• A set is “countable” if it is:
  • Finite
  • Or, there exists a bijection between the set and the natural numbers
    • This set is then considered to have the same size as the set of natural numbers
    • This is called “countably infinite”
Exercise: Which set is larger?

• The set of:
  • Natural numbers, or
  • Even numbers?

• They are the same size! Both are countably infinite
  • Bijection:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n) = 2n$</th>
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<tbody>
<tr>
<td>1</td>
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<td>2</td>
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Exercise: Which set is larger?

• The set of:
  • Natural numbers $\mathcal{N}$, or
  • Positive rational numbers? $Q = \left\{ \frac{m}{n} \mid m, n \in \mathcal{N} \right\}$
• They are the same size! Both are countably infinite.
Exercise: Which set is larger?

• The set of:
  • Natural numbers $\mathbb{N}$, or
  • Positive rational numbers? $\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{N} \right\}$

• They are the same size! Both are countably infinite

Another mapping: (is a bijection)
Exercise: Which set is larger?

• The set of:
  • Natural numbers, or $\mathbb{N}$
  • Real numbers? $\mathbb{R}$
• There are more real numbers. It is uncountably infinite.

Proof, by contradiction:
• Assume a bijection between natural and real numbers exists.
  • This means that every real number should get mapped to.
• But we show that in any given mapping, e.g.:
  • Some real number is not mapped to ...
  • E.g., any number that has different digits at each position:
    $$x = 0.4641\ldots$$
• This number is cannot included in mapping
• Contradiction!
Georg Cantor

• Invented set theory

• Came up with countable infinity in 1873

• And uncountability:
  • And how to show uncountability with “diagonalization” technique
Diagonalization with Turing Machines

### Diagonal: Result of Giving a TM its own Encoding as Input

<table>
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<tr>
<th></th>
<th>$\langle M_1 \rangle$</th>
<th>$\langle M_2 \rangle$</th>
<th>$\langle M_3 \rangle$</th>
<th>$\langle M_4 \rangle$</th>
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<tbody>
<tr>
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<td>$M_3$</td>
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<td>accept</td>
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</table>

- **All TMs**
- **All TM Encodings**
- **opposites**
- **Try to construct “opposite” TM**

**TM $D$ can’t exist!**

**What should happen here?**
Thm: $A_{TM}$ is undecidable

$$A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

Proof by contradiction:

• Assume $A_{TM}$ is decidable. Then there exists a decider:

$$H(\langle M, w \rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

• If $H$ exists, then we can create:

$$D = \text{“On input } \langle M \rangle, \text{ where } M \text{ is a TM:}$$

1. Run $H$ on input $\langle M, \langle M \rangle \rangle$.
2. Output the opposite of what $H$ outputs. That is, if $H$ accepts, reject; and if $H$ rejects, accept.”

• But $D$ does not exist! Contradiction! So assumption is false.
Turing Unrecognizable?

Is there anything out here?

$A_{TM}$

Turing-recognizable

decidable

context-free

regular
Thm: Some langs are not Turing-recognizable

Proof: requires 2 lemmas

• **Lemma 1**: The set of all languages is **uncountable**
  • **Proof**: Show there is a bijection with another uncountable set ...
    • ... The set of all infinite binary sequences

• **Lemma 2**: The set of all TMs is **countable**

• Therefore, some language is not recognized by a TM
Mapping a Language to a Binary Sequence

$$\Sigma^* = \{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots \}$$

$$A = \{ 0, 00, 01, 000, 001, \ldots \}$$

$$\chi_A = 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ \ldots$$

Each digit represents one possible string:
- 1 if lang has that string,
- 0 otherwise
**Thm:** Some langs are not Turing-recognizable

**Proof:** requires 2 lemmas

- **Lemma 1:** The set of all languages is uncountable
  - **Proof:** Show there is a bijection with another uncountable set ...
    - ... The set of all infinite binary sequences
      ➢ Now just prove set of infinite binary sequences is uncountable (hw8)

- **Lemma 2:** The set of all TMs is countable
  - Because every TM $M$ can be encoded as a string $<M>$
  - And set of all strings is countable

- Therefore, some language is not recognized by a TM
Co-Turing-Recognizability

- A language is **co-Turing-recognizable** if ...
- ... it is the complement of a Turing-recognizable language.
**Thm**: Decidable $\iff$ Recognizable & co-Recognizable

=>$\quad$ If a language is decidable, then it is recognizable and co-recognizable

  • Decidable $\Rightarrow$ Recognizable:
    • A decider is a recognizer, bc decidable langs are a subset of recognizable langs
  • Decidable $\Rightarrow$ Co-Recognizable:
    • To create co-decider (which is also a co-recognizer) from a decider ...
    • ... switch reject/accept of all inputs

<=$\quad$ If a language is recognizable and co-recognizable, then it is decidable
Thm: Decidable $\iff$ Recognizable & co-Recognizable

$\Rightarrow$ If a language is decidable, then it is recognizable and co-recognizable
  - Decidable $\Rightarrow$ Recognizable:
    - A decider is a recognizer, bc decidable langs are a subset of recognizable langs
  - Decidable $\Rightarrow$ Co-Recognizable:
    - To create co-decider (which is also a co-recognizer) from a decider ...
    - ... switch reject/accept of all inputs

$\Leftarrow$ If a language is recognizable and co-recognizable, then it is decidable
  - Let $M_1$ = recognizer for the language,
  - And $M_2$ = recognizer for its complement
  - Decider $M$:
    - Run 1 step on $M_1$,
    - Run 1 step on $M_2$,
    - Repeat, until one machine accepts. If it’s $M_1$, accept. If it’s $M_2$, reject
  - One of $M_1$ or $M_2$ must accept and halt, so $M$ halts and is a decider
A Turing-unrecognizable language

• We’ve proved:
  \[ A_{\text{TM}} \text{ is Turing-recognizable} \]
  \[ A_{\text{TM}} \text{ is undecidable} \]

• So:
  \[ \overline{A}_{\text{TM}} \text{ is not Turing-recognizable} \]

• Because: recognizable & co-recognizable implies decidable
Is there anything out here?
Next time: Easier Undecidability Proofs!

• We proved \( A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \} \) undecidable by ...

• ... showing that it’s decider could be used to implement an impossible “D” decider.
  • This was hard (need diagonalization)

• In other words, we reduced \( A_{TM} \) to the “D” problem.

• But now we can just reduce things to \( A_{TM} \): much easier!
Next time: The Halting Problem

\[ \text{HALT}_\text{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ halts on input } w \} \]

Thm: \( \text{HALT}_\text{TM} \) is undecidable

Proof, by contradiction:
- Assume \( \text{HALT}_\text{TM} \) has decider \( R \)
- Use it to create decider for \( A_{\text{TM}} \):
  - ...
- But \( A_{\text{TM}} \) is undecidable!
Check-in Quiz 3/31

On gradescope