Context-Free Languages (CFLs)

Monday, March 6, 2023
Announcements

• HW 4 in
  • due Sun 3/5 11:59pm EST

• HW 5 out
  • due Sun 3/19 11:59pm EST
  • (after Spring Break!)

Quiz Preview

• What do we call the class of languages that are generated by a CFG?
Last Time:

Pumping lemma: If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Let $B$ be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that $B$ is not regular. The proof is by contradiction.

• Assume: language $B$ is regular

• So it must satisfy the Pumping Lemma:
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- **Assume:** language $B$ is regular

- So it must satisfy the **Pumping Lemma**:
  - All strings $\geq$ length $p$ ...
  - ... can be split into some $xyz$ ... where $y$ is “pumpable”

- Get **contradiction** by finding **counterexample**: a not “pumpable string $\geq$ length $p$: $0p1p$
  - Must show string cannot be pumped for all possible splittings into $xyz$
  - Use pumping lemma condition #3 to eliminate some cases

- **Therefore, $B$ is not regular**
  - (This is the **contrapositive** of the Pumping Lemma)
  - This is a **contradiction** of the assumption!
Let $B$ be the language $\{0^n1^n \mid n \geq 0\}$. We use the pumping lemma to prove that $B$ is not regular. The proof is by contradiction.

If this language is not regular, then what is it???

Maybe? ... a context-free language (CFL)?
A Context-Free Grammar (CFG)

Top variable is: 
**Start variable**

Variables
(a.k.a., **nonterminals**) on RHS of rules

**terminals**

- $A \rightarrow 0A1$
- $A \rightarrow B$
- $B \rightarrow \#$

**Substitution rules** (a.k.a., **productions**)

**terminals** (analogous to a DFA's alphabet)
A context-free grammar (CFG)

Grammar $G_1 = (V, \Sigma, R, S)$

- **$R$** is this set of rules (mappings):
  - Top variable is: **Start variable**
  - **Variables** (a.k.a., **nonterminals**)
  - **Branch**
    - $A \rightarrow 0A1$
    - $A \rightarrow B$
    - $B \rightarrow \#$

- **Terminals** (analogous to a DFA's alphabet)

**CFG Practical Application:**
Used to describe programming language syntax!

A context-free grammar is a 4-tuple $(V, \Sigma, R, S)$, where

1. $V$ is a finite set called the **variables**, 
2. $\Sigma$ is a finite set, disjoint from $V$, called the **terminals**, 
3. $R$ is a finite set of **rules**, with each rule being a variable and a string of variables and terminals, and 
4. $S \in V$ is the start variable.
Java Syntax: Described with CFGs

Chapter 2. Grammars

This chapter describes the context-free grammars used in this specification to define the lexical and syntactic structure of a program.

2.1. Context-Free Grammars

A context-free grammar consists of a number of productions. Each production has an abstract symbol called a nonterminal as its left-hand side, and a sequence of one or more nonterminal and terminal symbols as its right-hand side. For each grammar, the terminal symbols are drawn from a specified alphabet.

Starting from a sentence consisting of a single distinguished nonterminal, called the goal symbol, a given context-free grammar specifies a language, namely, the set of possible sequences of terminal symbols that can result from repeatedly replacing any nonterminal in the sequence with a right-hand side of a production for which the nonterminal is the left-hand side.

2.2. The Lexical Grammar

A lexical grammar for the Java programming language is given in §3. This grammar has as its terminal symbols the characters of the Unicode character set. It defines a set of productions, starting from the goal symbol input (§3.5), that describe how sequences of Unicode characters (§3.1) are translated into a sequence of input elements (§3.6).
## Analogies

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*thm*  
*def*
10. Full Grammar specification

This is the full Python grammar, as it is read by the parser generator and used to parse Python source files:

```python
# Grammar for Python
# NOTE WELL: You should also follow all the steps listed at
# https://devguide.python.org/grammar/
# Start symbols for the grammar:
#     single_input is a single interactive statement;
#     file_input is a module or sequence of commands read from an input file;
#     eval_input is the input for the eval() functions.
#     func_type_input is a PEP 484 Python 2 function type comment
# NB: compound_stmt in single_input is followed by extra NEWLINE!
# NB: due to the way TYPE_COMMENT is tokenized it will always be followed by a NEWLINE
single_input:  NEWLINE  |  simple_stmt  |  compound_stmt  NEWLINE
file_input:   (NEWLINE  |  stmt)*  ENDMARKER
eval_input:   testlist  NEWLINE*  ENDMARKER
```

https://docs.python.org/3/reference/grammar.html
Many Other Language  (partially)  

Python Syntax: Described with a CFG

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Generating Strings with a CFG

Definition:
A CFG describes a context-free language! but what strings are in the language?

Strings in CFG’s language = all possible generated / derived strings

A CFG generates a string, by repeatedly applying substitution rules:

\[ L(G_1) = \{0^n#1^n \mid n \geq 0\} \]

Start variable: A

After applying 1st rule: 0A1

1st rule again: 00A11

1st rule again: 000A111

Use 2nd rule: 000B111

Use last rule: 000#111
Derivations: Formally

Let \( G = (V, \Sigma, R, S) \)

**Single-step**

\[
\alpha A \beta \xrightarrow{G} \alpha \gamma \beta
\]

Where:

- \( \alpha, \beta \in (V \cup \Sigma)^* \)
- \( A \in V \)
- \( A \rightarrow \gamma \in R \)

**Extended Derivation**

**Base case:** \( \alpha \xrightarrow{G} \alpha \) (0 steps)

**Recursive case:** (multistep)

- If \( \alpha \xrightarrow{G} \beta \) and \( \beta \xrightarrow{G} \gamma \)
- Single step
- Then: \( \alpha \xrightarrow{G} \gamma \)
Formal Definition of a CFL

A context-free grammar is a 4-tuple \((V, \Sigma, R, S)\), where

1. \(V\) is a finite set called the **variables**,  
2. \(\Sigma\) is a finite set, disjoint from \(V\), called the **terminals**,  
3. \(R\) is a finite set of **rules**, with each rule being a variable and a string of variables and terminals, and  
4. \(S \in V\) is the start variable.

\[ G = (V, \Sigma, R, S) \]

\[ L(G) = \{ w \in \Sigma^* \mid S \xrightarrow{G} w \} \]

Any language that can be generated by some context-free grammar is called a **context-free language**.
Flashback: \{0^n1^n \mid n \geq 0\}

- Pumping Lemma says it’s not a regular language
- It’s a context-free language!
  - Proof?
  - Come up with CFG describing it ...
  - \textbf{Hint}: It’s similar to:

\[
\begin{align*}
A & \rightarrow 0A1 \\
A & \rightarrow B \\
B & \rightarrow \# \varepsilon
\end{align*}
\]

$L(G_1)$ is $\{0^n\#1^n \mid n \geq 0\}$

Statements and Justifications?
Proof: \( L = \{ 0^n 1^n \mid n \geq 0 \} \) is a CFL

**Statements**

1. If a CFG describes a language, then it is a CFL

2. CFG \( G_1 \) describes \( L \)
   
   \[
   \begin{align*}
   A & \rightarrow 0A1 \\
   A & \rightarrow B \\
   B & \rightarrow \epsilon
   \end{align*}
   \]

3. \( L = \{ 0^n 1^n \mid n \geq 0 \} \) is a CFL

**Justifications**

1. Definition of CFL

2. (Did you come up with examples???)

3. By Statements #1 and #2
A String Can Have Multiple Derivations

\[
\begin{align*}
\langle \text{EXPR} \rangle & \rightarrow \langle \text{EXPR} \rangle + \langle \text{TERM} \rangle | \langle \text{TERM} \rangle \\
\langle \text{TERM} \rangle & \rightarrow \langle \text{TERM} \rangle \times \langle \text{FACTOR} \rangle | \langle \text{FACTOR} \rangle \\
\langle \text{FACTOR} \rangle & \rightarrow ( \langle \text{EXPR} \rangle ) | a
\end{align*}
\]

Shorthand for multiple rules with same nonterminal LHS

Want to generate this string: \( a + a \times a \)

- \( \text{EXPR} \Rightarrow \)
- \( \text{EXPR} + \text{TERM} \Rightarrow \)
- \( \text{EXPR} + \text{TERM} \times \text{FACTOR} \Rightarrow \)
- \( \text{EXPR} + \text{TERM} \times a \Rightarrow \)
- \( \ldots \)

RIGHTMOST DERIVATION

- \( \text{EXPR} \Rightarrow \)
- \( \text{EXPR} + \text{TERM} \Rightarrow \)
- \( \text{TERM} + \text{TERM} \Rightarrow \)
- \( \text{FACTOR} + \text{TERM} \Rightarrow \)
- \( a + \text{TERM} \)
- \( \ldots \)

LEFTMOST DERIVATION
Derivations and Parse Trees

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

A derivation may also be represented as a parse tree
Multiple Derivations, Single Parse Tree

- \( \text{EXPR} \rightarrow \)
- \( \text{EXPR} + \text{TERM} \rightarrow \)
- \( \text{TERM} + \text{TERM} \rightarrow \)
- \( \text{FACTOR} + \text{TERM} \rightarrow \)
- \( a + \text{TERM} \)
- ...

A parse tree represents a CFG computation ... like a sequence of states represents a DFA computation

A Parse Tree gives “meaning” to a string
Ambiguity grammar $G_5$:

$$
\langle \text{EXPR} \rangle \rightarrow \langle \text{EXPR} \rangle + \langle \text{EXPR} \rangle \mid \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle \mid (\langle \text{EXPR} \rangle) \mid a
$$

Same string, different derivation, and different parse tree!

So this string has two meanings!
A string $w$ is derived \textit{ambiguously} in context-free grammar $G$ if it has two or more different leftmost derivations. Grammar $G$ is \textit{ambiguous} if it generates some string ambiguously.

An ambiguous grammar can give a string \textit{multiple meanings}, ie represent \textit{two different computations}! (why is this \textit{bad}?)
Real-life Ambiguity ("Dangling" else)

- What is the result of this C program?

```c
if (1) if (0) printf("a"); else printf("2");
if (1)
  if (0)
    printf("a");
else
  printf("2");
```

```c
if (1)
  if (0)
    printf("a");
else
  printf("2");
```

This string has 2 parsings, and thus 2 meanings!

Ambiguous grammars are confusing. A computation on a string should ideally have only one result.

Thus in practice, we typically focus on the unambiguous subset of CFGs (CFLs) (more on this later)

Problem is, there's no easy way to create an unambiguous grammar (it's up to language designers to "be careful")
Designing Grammars : Basics

1. Think about what you want to “link” together

• E.g., $0^n1^n$
  • $A \rightarrow 0A1$
  • # 0s and # 1s are “linked”

• E.g., XML
  • ELEMENT $\rightarrow <TAG>CONTENT</TAG>$
  • Start and end tags are “linked”

2. Start with small grammars and then combine (just like FSMs)
Designing Grammars: Building Up

• Start with small grammars and then combine (just like FSMs)
  
  • To create a grammar for the language \( \{0^n1^n | n \geq 0\} \cup \{1^n0^n | n \geq 0\} \):
    
    \[
    S_1 \rightarrow 0S_11 | \varepsilon
    \]
    
    • Then create grammar for lang \( \{1^n0^n | n \geq 0\} \):
    
    \[
    S_2 \rightarrow 1S_20 | \varepsilon
    \]
    
    • Then combine:
    
    \[
    S \rightarrow S_1 | S_2 \\
    S_1 \rightarrow 0S_11 | \varepsilon \\
    S_2 \rightarrow 1S_20 | \varepsilon
    \]

  New start variable and rule combines two smaller grammars

  “|” = “or” = union (combines 2 rules with same left side)
(Closed) Operations on CFLs?

• Start with small grammars and then combine (just like FSMs)

• “Or”: \[ S' \rightarrow S_1 \mid S_2 \]

• “Concatenate”: \[ S' \rightarrow S_1 S_2 \]

• “Repetition”: \[ S' \rightarrow S' S_1 \mid \varepsilon \]
In-class Example: Designing grammars

alphabet $\Sigma$ is \{0,1\}

\[
\{w | \text{w starts and ends with the same symbol}\}
\]

1) come up with examples: In the language: 010, 101, 11011 1, 0 ?
Not in the language: 10, 01, 110 $\varepsilon$ ?

2) Create CFG:

\[
S \rightarrow 0C'0 \mid 1C'1 \mid 0 \mid 1
\]

“string starts/ends with same symbol, middle can be anything”

\[
C' \rightarrow C'C \mid \varepsilon
\]

“middle: all possible terminals, repeated (ie, all possible strings)”

\[
C \rightarrow 0 \mid 1
\]

“all possible terminals”

3) Check CFG: generates examples in the language; does not generate examples not in language
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**DIFERENCE:**

| A Regular Lang is defined with a FSM | A CFL is defined with a CFG |

**Proved:** Reg Expr $\Leftrightarrow$ Reg Lang

**Must prove:** PDA $\Leftrightarrow$ CFL
Check-in Quiz 3/6

On gradescope