CS420
Reducibility
Wednesday, April 24, 2024
Announcements

• HW 9 in
  • Due Wed 4/24 12pm noon

• HW 10 out
  • Due Wed 5/1 12pm noon
Thm: $A_{TM}$ is undecidable

Proof by contradiction:

1. Assume $A_{TM}$ is decidable. So there exists a decider $H$ for it:

$$H(\langle M, w \rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

2. Use $H$ in another TM ... the impossible “opposite” machine:

   $D = \text{"On input } \langle M \rangle, \text{ where } M \text{ is a TM:}\$
   
   1. Run $H$ on input $\langle M, \langle M \rangle \rangle$. $H$ computes: $M$'s result with $\langle M \rangle$ as input
   
   2. Output the opposite of what $H$ outputs. That is, if $H$ accepts, reject; and if $H$ rejects, accept.” $D$ returns opposite of $H$
Thm: $A_{TM}$ is undecidable

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$

Proof by contradiction:

1. Assume $A_{TM}$ is decidable. So there exists a decider $H$ for it:

   $H(\langle M, w \rangle) =\begin{cases} accept & \text{if } M \text{ accepts } w \\ reject & \text{if } M \text{ does not accept } w \end{cases}$

2. Use $H$ in another TM ... the impossible “opposite” machine:

   $D = \text{“On input } \langle M \rangle \text{, where } M \text{ is a TM:}\
   1. \text{ Run } H \text{ on input } \langle M, \langle M \rangle \rangle.\
   2. \text{ Output the opposite of what } H \text{ outputs. That is, if } H \text{ accepts, reject; and if } H \text{ rejects, accept.”}$

3. So $D$ does not exist! **Contradiction**! So the assumption is false.
Easier Undecidability Proofs

• We proved \( A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \} \) undecidable ...

• ... by contradiction:
  • Use hypothetical \( A_{TM} \) decider to create an impossible decider “D”!
  • **Reduce “D” problem to \( A_{TM} \)**

• Step # 1: coming up with “D” --- **hard**!
  • Need to invent **diagonalization**

• Step # 2: **Reduce “D” problem to \( A_{TM} \)** --- **easier**!

• From now on: undecidability proofs only need step # 2!
  • And we now have two “impossible” problems to choose from
The Halting Problem

\[
\text{HALT}_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ halts on input } w \}\]

**Thm:** \(\text{HALT}_{TM}\) is undecidable

**Proof**, by contradiction:

- **Assume:** \(\text{HALT}_{TM}\) has **decider** \(R\); use it to create decider for \(A_{TM}\);

Examples Table(s) are critical for these kinds of problems!

Let \(\langle M, w \rangle\) be a string where:
- \(M\) is some TM and
- \(w\) is some string

<table>
<thead>
<tr>
<th>String</th>
<th>(M) on (w)</th>
<th>(R) on (\langle M, w \rangle)</th>
<th>In lang (\text{HALT}_{TM})?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\langle M, w \rangle)</td>
<td>(halt and) Accept</td>
<td>Accept</td>
<td>Yes</td>
</tr>
<tr>
<td>(\langle M, w \rangle)</td>
<td>(halt and) Reject</td>
<td>Accept</td>
<td>Yes</td>
</tr>
<tr>
<td>(\langle M, w \rangle)</td>
<td>Loop</td>
<td>Reject</td>
<td>No</td>
</tr>
</tbody>
</table>
The Halting Problem

**Thm:** $HALT_{TM}$ is undecidable

**Proof,** by contradiction:

• **Assume:** $HALT_{TM}$ has decider $R$; use it to create decider for $A_{TM}$:

$$A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$$

• ... 

• But $A_{TM}$ is undecidable and has no decider!
The Halting Problem

\[ HALT_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ halts on input } w \} \]

**Thm:** \( HALT_{TM} \) is undecidable

**Proof, by contradiction:**

- **Assume:** \( HALT_{TM} \) has decider \( R \); use it to create decider for \( A_{TM} \):

\[ A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \} \]

\( S = \) “On input \( \langle M, w \rangle \), an encoding of a TM \( M \) and a string \( w \):

1. Run TM \( R \) on input \( \langle M, w \rangle \).
2. If \( R \) rejects, reject.
3. If \( R \) accepts, simulate \( M \) on \( w \) until it halts.
4. If \( M \) has accepted, accept; if \( M \) has rejected, reject.”

*Termination argument:*

**Step 1:** \( R \) is a decider so always halts

**Step 3:** \( M \) always halts because \( R \) said so

*Examples Table??*
The Halting Problem

**Thm**: $HALT_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ halts on input } w \}$

<table>
<thead>
<tr>
<th>String</th>
<th>$M$ on $w$</th>
<th>$HALT_{TM}$ decider $R$ on $\langle M, w \rangle$</th>
<th>$A_{TM}$ decider $S$ on $\langle M, w \rangle$</th>
<th>In lang $A_{TM}$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle M, w \rangle$</td>
<td>Accept</td>
<td>Accept</td>
<td>Accept</td>
<td>Yes</td>
</tr>
<tr>
<td>$\langle M, w \rangle$</td>
<td>Reject</td>
<td>Accept</td>
<td>Reject</td>
<td>No</td>
</tr>
<tr>
<td>$\langle M, w \rangle$</td>
<td>Loop</td>
<td>Reject</td>
<td>Reject</td>
<td>No</td>
</tr>
</tbody>
</table>

Let $\langle M, w \rangle$ be a string where:
- $M$ is some TM and
- $w$ is some string

$A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}$

**Example Table for $A_{TM}$ decider $S$**

---

$s = \text{"On input } \langle M, w \rangle, \text{ an encoding of a TM } M \text{ and a string } w:\n1. \text{ Run } TM R \text{ on input } \langle M, w \rangle.\n2. \text{ If } R \text{ rejects, reject.}\n3. \text{ If } R \text{ accepts, simulate } M \text{ on } w \text{ until it halts.}\n4. \text{ If } M \text{ has accepted, accept; if } M \text{ has rejected, reject.\"} $
The Halting Problem

**Thm:** $\text{HALT}_{TM}$ is undecidable

**Proof**, by contradiction:

- **Assume:** $\text{HALT}_{TM}$ has *decider* $R$; use it to create decider for $A_{TM}$:

  $$S = \text{"On input } \langle M, w \rangle, \text{ an encoding of a TM } M \text{ and a string } w:\n  1. \text{ Run TM } R \text{ on input } \langle M, w \rangle. 
  2. \text{ If } R \text{ rejects, reject.} 
  3. \text{ If } R \text{ accepts, simulate } M \text{ on } w \text{ until it halts.} 
  4. \text{ If } M \text{ has accepted, accept; if } M \text{ has rejected, reject."}$$

- But $A_{TM}$ is undecidable *(has no decider)*! I.e., this decider does not exist!
- So $\text{HALT}_{TM}$ is also undecidable!
The Halting Problem ... As Statements / Justifications

Proof by contradiction

Statements
1. $HALT_{TM}$ is decidable
2. $HALT_{TM}$ has decider $R$
3. Construct decider $S$ using $R$ (“see below”)
4. Decider $S$ decides $A_{TM}$
5. $A_{TM}$ is undecidable (i.e., it has no decider)
6. $HALT_{TM}$ is undecidable

Justifications
1. Opposite of statement to prove
2. Definition of decidable langs
3. Definition of TMs and deciders (incl. termination argument)
4. See Examples Table
5. Theorem from last lecture (Sipser Theorem 4.11)
6. Contradiction of Stmts #4 & #5

$HALT_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \}$
Interlude: Reducing from $\text{HALT}^\text{TM}$

A practical thought experiment ...  
... about compiler optimizations

Your compiler changes your program!

If $\text{TRUE}$ then $\text{A}$ else $\text{B}$  $\rightarrow$  $\text{A}$

$1 + 2 + 3$  $\rightarrow$  $6$
Optimization - docs

- **-00**
  - No optimization, faster compilation time, better for debugging builds.
- **-02**
- **-03**
  - Higher level of optimization. Slower compile-time, better for production builds.
- **-OFast**
  - Enables higher level of optimization than (-03). It enables lots of flags as can be seen in the source code (-ffloat-store, -ffast-math, -ffinite-math-only, -O3 ...)
- **-finline-functions**
- **-m64**
- **-funroll-loops**
- **-fvectorize**
- **-fprofile-generate**

**Types of optimization**

Techniques used in optimization can be broken up among various scopes which can affect anything from a single statement to the entire program. Generally speaking, locally scoped techniques are easier to implement than global ones but result in smaller gains. Some examples of scopes include:

- **Peephole optimizations**
  - These are usually performed late in the compilation process after machine code has been generated. This form of optimization examines a few adjacent instructions (like looking through a peephole at the code) to see whether they can be replaced by a single instruction or a shorter sequence of instructions. For instance, a multiplication of a value by 2 might be more efficiently executed by left-shifting the value or by adding the value to itself. This example is also an instance of strength reduction.

- **Loop optimizations**
  - These act on the statements which make up a loop, such as a for loop, for example loop-invariant code motion. Loop optimizations can have a significant impact because many programs spend a large percentage of their time inside loops.

- **Prescient store optimizations**
  - These allow store operations to occur earlier than would otherwise be permitted in the context of threads and locks. The process needs some way of knowing ahead of time what value will be stored by the assignment that it should have followed. The purpose of this relaxation is to allow compiler optimization to perform certain kinds of code rearrangement that preserve the semantics of properly synchronized programs.

- **Interprocedural, whole-program or link-time optimization**
  - These analyze all of a program's source code. The greater quantity of information extracted means that optimizations can be more effective compared to when they only have access to local information, i.e., within a single function. This kind of optimization can also allow new techniques to be performed. For instance, function inlining, where a call to a function is replaced by a copy of the function body.

- **Machine code optimization and object code optimizer**
  - These analyze the executable task image of the program after all of an executable machine code has been linked. Some of the techniques that can be applied in a more limited scope, such as macro compression which saves space by collapsing common sequences of instructions, are more effective when the entire executable task image is available for analysis.
The Optimal Optimizing Compiler

**Thm:** The Optimal (C++) Optimizing Compiler does not exist

**Proof**, by contradiction:

**Assume:** $OPT$ is the Perfect Optimizing Compiler

Use it to create $HALT_{TM}$ decider (accepts $<M,w>$ if $M$ halts with $w$, else rejects):

$S = \text{On input } <M, w>, \text{ where } M \text{ is C++ program and } w \text{ is string:}$

- If $OPT(M) = \text{for} (;;)$
  - a) Then **Reject**
  - b) Else **Accept**

---

**“Full Employment” Theorem**

In computer science and mathematics, a **full employment theorem** is a term used, often humorously, to refer to a theorem which states that no algorithm can optimally perform a particular task done by some class of professionals. The name arises because such a theorem ensures that there is endless scope to keep discovering new techniques to improve the way at least some specific task is done.

For example, the **full employment theorem for compiler writers** states that there is no such thing as a provably perfect size-optimizing compiler, as such a proof for the compiler would have to detect non-terminating computations and reduce them to a one-instruction infinite loop. Thus, the existence of a provably perfect size-optimizing compiler would imply a solution to the halting problem, which cannot exist. This also implies that there may always be a better compiler since the proof that one has the best compiler cannot exist. Therefore, compiler writers will always be able to speculate that they have something to improve.
Summary: The Limits of Algorithms

- $A_{DFA} = \{ \langle B, w \rangle | B \text{ is a DFA that accepts input string } w \}$
  - Decidable

- $A_{CFG} = \{ \langle G, w \rangle | G \text{ is a CFG that generates string } w \}$
  - Decidable

- $A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}$
  - Undecidable

- $HALT_{TM} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ halts on input } w \}$
  - Undecidable

It’s straightforward to use hypothetical $HALT_{TM}$ decider to create $A_{TM}$ decider
Summary: The Limits of Algorithms

- $A_{DFA} = \{ \langle B, w \rangle | \ B \text{ is a DFA that accepts input string } w \}$
  Decidable
- $A_{CFG} = \{ \langle G, w \rangle | \ G \text{ is a CFG that generates string } w \}$
  Decidable
- $A_{TM} = \{ \langle M, w \rangle | \ M \text{ is a TM and } M \text{ accepts } w \}$
  Undecidable
- $HALT_{TM} = \{ \langle M, w \rangle | \ M \text{ is a TM and } M \text{ halts on input } w \}$
  Undecidable
- $E_{DFA} = \{ \langle A \rangle | \ A \text{ is a DFA and } L(A) = \emptyset \}$
  Decidable
- $E_{CFG} = \{ \langle G \rangle | \ G \text{ is a CFG and } L(G) = \emptyset \}$
  Decidable
- $E_{TM} = \{ \langle M \rangle | \ M \text{ is a TM and } L(M) = \emptyset \}$
  Undecidable

How can we use a hypothetical $E_{TM}$ decider to create $A_{TM}$ or $HALT_{TM}$ decider?
Thm: $E_{TM}$ is undecidable

Proof, by contradiction:

- Assume $E_{TM}$ has decider $R$; use it to create decider for $A_{TM}$:

$$S = \text{“On input } \langle M, w \rangle, \text{ an encoding of a TM } M \text{ and a string } w:}$$

- Run $R$ on input $\langle M \rangle$
- If $R$ accepts, reject (because it means $\langle M \rangle$ doesn’t accept anything)
- If $R$ rejects, then ??? ($\langle M \rangle$ accepts something, but is it $w$???)

<table>
<thead>
<tr>
<th>String</th>
<th>$M$ on $w$</th>
<th>$R$ on $\langle M \rangle$</th>
<th>$S$ on $\langle M, w \rangle$</th>
<th>In lang $A_{TM}$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle M, w \rangle$</td>
<td>Accept</td>
<td>??</td>
<td>Accept</td>
<td>Yes</td>
</tr>
<tr>
<td>$\langle M, w \rangle$</td>
<td>Reject</td>
<td>??</td>
<td>Reject</td>
<td>No</td>
</tr>
<tr>
<td>$\langle M, w \rangle$</td>
<td>Loop</td>
<td>??</td>
<td>Reject</td>
<td>No</td>
</tr>
</tbody>
</table>

Examples Table(s) are critical here!

Undecidability Proof Technique #2
Reducibility: Modifying the TM

**Thm:** $E_{TM}$ is undecidable

**Proof,** by contradiction:

- Assume $E_{TM}$ has *decider* $R$; use it to create *decider* for $A_{TM}$:
  
  $S = \text{"On input } \langle M, w \rangle \text{, an encoding of a TM } M \text{ and a string } w:\$
  
  - Run $R$ on input $\langle M \rangle$
  - If $R$ accepts, *reject* (because it means $\langle M \rangle$ doesn’t accept anything)
  - If $R$ rejects, then ??? ($\langle M \rangle$ accepts something, but is it $w$???)

- **Idea:** Wrap $\langle M \rangle$ in a new TM that can only *(maybe)* accept $w$

  $M_1 = \text{"On input } x:\$
  
  1. If $x \neq w$, reject.
  2. If $x = w$, run $M$ on input $w$ and accept if $M$ does.

  $L(M_1)$ depends on $M$ and $w$!
  If $M$ accepts $w$, $L(M_1) = \{w\}$
  else $L(M_1) = \{}$

  Input is $w$, maybe accept

  Input not $w$, always reject

  $M_1$ accepts $w$ if $M$ does
Reducibility: Modifying the TM

Thm: $E_{TM}$ is undecidable

Proof, by contradiction:

- Assume $E_{TM}$ has decider $R$; use it to create decider for $A_{TM}$:
  
  $S = \text{"On input } \langle M, w \rangle, \text{ an encoding of a TM } M \text{ and a string } w:\$

  
<table>
<thead>
<tr>
<th>String $x$</th>
<th>$M$ on $w$</th>
<th>$R$ on $\langle M_1 \rangle$</th>
<th>$M_1$ on $x$</th>
<th>In $\text{lang } {w} \cap L(M)$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>Accept</td>
<td>Reject</td>
<td>Accept</td>
<td>Yes (lang = ${w}$)</td>
</tr>
<tr>
<td>$w$</td>
<td>Reject</td>
<td>Accept</td>
<td>Reject</td>
<td>No (lang = ${}$)</td>
</tr>
<tr>
<td>not $w$</td>
<td>-</td>
<td>-</td>
<td>Reject</td>
<td>No</td>
</tr>
</tbody>
</table>

- **Idea:** Wrap $\langle M \rangle$ in a new TM that can only (maybe) accept $w$.

  $M_1 = \text{"On input } x:\$
  
  1. If $x \neq w$, reject.
  2. If $x = w$, run $M$ on input $w$ and accept if $M$ does."
Thm: $E_{\text{TM}}$ is undecidable

Proof, by contradiction:

• Assume $E_{\text{TM}}$ has decider $R$; use it to create decider for $A_{\text{TM}}$:

  $S =$ “On input $\langle M, w \rangle$, an encoding of a TM $M$ and a string $w$:
  
  1. Run $R$ on input $\langle M \rangle$.
  2. If $R$ accepts, reject (because it means $\langle M \rangle$ doesn’t accept $w$).
  3. If $R$ rejects, then accept ($\langle M \rangle$ accepts $w$).

• Idea: Wrap $\langle M \rangle$ in a new TM that can only (maybe) accept $w$.

  $M_1 =$ “On input $x$:
  1. If $x \neq w$, reject.
  2. If $x = w$, run $M$ on input $w$ and accept if $M$ does.”

$L(M_1)$ depends on $M$ and $w$! If $M$ accepts $w$, $L(M_1) = \{w\}$ else $L(M_1) = \{}$
Reducibility: Modifying the TM

Thm: $E_{TM}$ is undecidable

Proof, by contradiction:

- Assume $E_{TM}$ has **decider** $R$; use it to create **decider** for $A_{TM}$:

  $S = \text{"On input } \langle M, w \rangle, \text{ an encoding of a TM } M \text{ and a string } w:\$
  
  - Run $R$ on input $\langle M_1 \rangle$
  - If $R$ accepts, reject (because it means $\langle M \rangle$ doesn’t accept $w$)
  - If $R$ rejects, then accept ($\langle M \rangle$ accepts $w$)

- **Idea:** Wrap $\langle M \rangle$ in a new TM that **can only** (maybe) accept $w$:

  $M_1 = \text{"On input } x:\$
  
  1. If $x \neq w$, reject.
  2. If $x = w$, run $M$ on input $w$ and accept if $M$ does."
Summary: The Limits of Algorithms

- \( A_{DFA} = \{\langle B, w \rangle \mid B \text{ is a DFA that accepts input string } w \} \) Decidable
- \( A_{CFG} = \{\langle G, w \rangle \mid G \text{ is a CFG that generates string } w \} \) Decidable
- \( A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \} \) Undecidable
- \( E_{DFA} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset \} \) Decidable
- \( E_{CFG} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset \} \) Decidable
- \( E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \} \) Undecidable
- \( EQ_{DFA} = \{\langle A, B \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B) \} \) Decidable
- \( EQ_{CFG} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H) \} \) Undecidable
- \( EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2) \} \) Undecidable
Reduce to something else: $EQ_{TM}$ is undecidable

$EQ_{TM} = \{\langle M_1, M_2 \rangle | M_1$ and $M_2$ are TMs and $L(M_1) = L(M_2)\}$

Proof, by contradiction:

- **Assume**: $EQ_{TM}$ has decider $R$; use it to create decider for $A_{TM}$:

  $E_{TM} = \{\langle M \rangle | M$ is a TM and $L(M) = \emptyset\}$

  \[
  S = \text{"On input } \langle M \rangle, \text{ where } M \text{ is a TM:
  
  1. Run } R \text{ on input } \langle M, M_1 \rangle, \text{ where } M_1 \text{ is a TM that rejects all inputs.}
  
  2. If } R \text{ accepts, accept; if } R \text{ rejects, reject."}
  \]
Reduce to something else: $EQ_{TM}$ is undecidable

$$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Proof, by contradiction:

- **Assume**: $EQ_{TM}$ has decider $R$; use it to create decider for $E_{TM}$:

  $$= \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

  $S = “\text{On input } \langle M \rangle, \text{ where } M \text{ is a TM:}$$

  1. Run $R$ on input $\langle M, M_1 \rangle$, where $M_1$ is a TM that rejects all inputs.
  2. If $R$ accepts, *accept*; if $R$ rejects, *reject*.”

- But $E_{TM}$ is undecidable!
Summary: Undecidability Proof Techniques

**Proof Technique #1:**
- Use hypothetical decider to implement impossible $A_{TM}$ decider
- Example Proof: $HALT_{TM} = \{\langle M, w \rangle | M \text{ is a TM and } M \text{ halts on input } w \}$

**Proof Technique #2:**
- Use hypothetical decider to implement impossible $A_{TM}$ decider
- But first modify the input $M$
- Example Proof: $E_{TM} = \{\langle M \rangle | M \text{ is a TM and } L(M) = \emptyset \}$

**Proof Technique #3:**
- Use hypothetical decider to implement non-$A_{TM}$ impossible decider
- Example Proof: $EQ_{TM} = \{\langle M_1, M_2 \rangle | M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$

Can also combine these techniques

Reduce
Summary: Decidability and Undecidability

- $A_{\text{DFA}} = \{ \langle B, w \rangle | B \text{ is a DFA that accepts input string } w \}$  
  Decidable
- $A_{\text{CFG}} = \{ \langle G, w \rangle | G \text{ is a CFG that generates string } w \}$  
  Decidable
- $A_{\text{TM}} = \{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}$  
  Undecidable
- $E_{\text{DFA}} = \{ \langle A \rangle | A \text{ is a DFA and } L(A) = \emptyset \}$  
  Decidable
- $E_{\text{CFG}} = \{ \langle G \rangle | G \text{ is a CFG and } L(G) = \emptyset \}$  
  Decidable
- $E_{\text{TM}} = \{ \langle M \rangle | M \text{ is a TM and } L(M) = \emptyset \}$  
  Undecidable
- $E_{Q_{\text{DFA}}} = \{ \langle A, B \rangle | A \text{ and } B \text{ are DFAs and } L(A) = L(B) \}$  
  Decidable
- $E_{Q_{\text{CFG}}} = \{ \langle G, H \rangle | G \text{ and } H \text{ are CFGs and } L(G) = L(H) \}$  
  Undecidable
- $E_{Q_{\text{TM}}} = \{ \langle M_1, M_2 \rangle | M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$  
  Undecidable
Also Undecidable ...

- $\text{REGULAR}_{TM} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}$
Thm: \( \text{REGULAR}_{TM} \) is undecidable

\[
\text{REGULAR}_{TM} = \{ \langle M \rangle | M \text{ is a TM and } L(M) \text{ is a regular language} \}
\]

Proof, by contradiction:

• **Assume:** \( \text{REGULAR}_{TM} \) has *decider* \( R \); use it to create *decider* for \( A_{TM} \):

\[
S = \text{“On input } \langle M, w \rangle, \text{ an encoding of a TM } M \text{ and a string } w:}
\]

  • **First, construct** \( M_2(M) \)
  • **Run** \( R \) **on input** \( \langle M_2 \rangle \)
  • **If** \( R \) **accepts,** accept; **if** \( R \) **rejects,** reject

Want: \( L(M_2) = \)

  • **regular,** if \( M \) **accepts** \( w \)
  • **nonregular,** if \( M \) **does not accept** \( w \)
Thm: $REGULAR_{TM}$ is undecidable (continued)

$REGULAR_{TM} = \{ \langle M \rangle \mid M$ is a TM and $L(M)$ is a regular language $\}$

$M_2 = \text{“On input } x:\text{ 1. If } x \text{ has the form } 0^n1^n, \text{ accept. 2. If } x \text{ does not have this form, run } M \text{ on input } w \text{ and accept if } M \text{ accepts } w.”}$

- If $M$ does not accept $w$, $M_2$ accepts all strings (regular lang)
- If $M$ accepts $w$, $M_2$ accepts this nonregular lang

Want: $L(M_2) =$
- regular, if $M$ accepts $w$
- nonregular, if $M$ does not accept $w$

Always accept strings $0^n1^n$
$L(M_2) =$ nonregular, so far

If $M$ accepts $w$, accept everything else, so $L(M_2) = \Sigma^* =$ regular
Also Undecidable ...

• $REGULAR_{TM} = \{< M > | M \text{ is a TM and } L(M) \text{ is a regular language}\}$

• $CONTEXTFREE_{TM} = \{< M > | M \text{ is a TM and } L(M) \text{ is a CFL}\}$

• $DECIDABLE_{TM} = \{< M > | M \text{ is a TM and } L(M) \text{ is a decidable language}\}$

• $FINITE_{TM} = \{< M > | M \text{ is a TM and } L(M) \text{ is a finite language}\}$

Seems like no algorithm can compute anything about the language of a Turing Machine, i.e., about the runtime behavior of programs ...
An Algorithm About Program Behavior?

```c
main()
{
    printf("hello, world\n");
}
```

**Write a program** that, given another program as its argument, returns **TRUE** if that argument prints "Hello, World!"

**TRUE**
Write a program that, given another program as its argument, returns TRUE if that argument prints "Hello, World!"

Fermat's Last Theorem (unknown for ~350 years, solved in 1990s)

```c
main()
{
    if (x^n + y^n == z^n, for any integer n > 2)
        printf("hello, world\n");
}
```
Also Undecidable ...

- $\text{REGULAR}_{TM} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}$

- $\text{CONTEXTFREE}_{TM} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a CFL}\}$

- $\text{DECIDABLE}_{TM} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a decidable language}\}$

- $\text{FINITE}_{TM} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a finite language}\}$

- ...

- $\text{ANYTHING}_{TM} = \{<M> \mid M \text{ is a TM and “… anything …” about } L(M)\}$

Seems like no algorithm can compute anything about the language of a Turing Machine, i.e., about the runtime behavior of programs ...

Rice’s Theorem
Rice’s Theorem: $\text{ANYTHING}_{\text{TM}}$ is Undecidable

$\text{ANYTHING}_{\text{TM}} = \{<M> \mid M \text{ is a TM and } \text{ ... anything ... about } L(M)\}$

- “... Anything ...”, more precisely:
  - For any $M_1, M_2$,
    - if $L(M_1) = L(M_2)$
    - then $M_1 \in \text{ANYTHING}_{\text{TM}} \iff M_2 \in \text{ANYTHING}_{\text{TM}}$

- Also, “... Anything ...” must be “non-trivial”:
  - $\text{ANYTHING}_{\text{TM}} \neq \{\}$
  - $\text{ANYTHING}_{\text{TM}} \neq \text{set of all TMs}$
Rice’s Theorem: \( \text{ANYTHING}_{\text{TM}} \) is Undecidable

\[ \text{ANYTHING}_{\text{TM}} = \{<M> \mid M \text{ is a TM and ... anything ... about } L(M)\} \]

Proof by contradiction

- **Assume** some language satisfying \( \text{ANYTHING}_{\text{TM}} \) has a decider \( R \).
  - Since \( \text{ANYTHING}_{\text{TM}} \) is non-trivial, then there exists \( M_{\text{ANY}} \in \text{ANYTHING}_{\text{TM}} \)
  - Where \( R \) accepts \( M_{\text{ANY}} \)

- **Use** \( R \) to create decider for \( A_{\text{TM}} \):

  **On input** \( <M, w> \):

  - **Create** \( M_w \):
    - \( M_w \) = on input \( x \):
      - Run \( M \) on \( w \)
      - If \( M \) rejects \( w \): reject \( x \)
      - If \( M \) accepts \( w \):
        - Run \( M_{\text{ANY}} \) on \( x \) and accept if it accepts, else reject
  - **Run** \( R \) on \( M_w \)
    - If it accepts, then \( M_w = M_{\text{ANY}} \), so \( M \) accepts \( w \), so accept
    - Else reject

- **Wait!** What if the TM that accepts nothing is in \( \text{ANYTHING}_{\text{TM}} \)?
  - These two cases must be different, (so \( R \) can distinguish when \( M \) accepts \( w \))

- **Proof still works!** Just use the complement of \( \text{ANYTHING}_{\text{TM}} \) instead!
Rice’s Theorem Implication

\[ \{<M> \mid M \text{ is a TM that installs malware}\} \]

Undecidable!
(by Rice’s Theorem)