A **nondeterministic finite automaton** is a 5-tuple \((Q, \Sigma, \delta, q_0, F)\), where

1. \(Q\) is a finite set of states,
2. \(\Sigma\) is a finite alphabet,
3. \(\delta: Q \times \Sigma_e \rightarrow \mathcal{P}(Q)\) is the transition function,
4. \(q_0 \in Q\) is the start state, and
5. \(F \subseteq Q\) is the set of accept states.

A **finite automaton** is a 5-tuple \((Q, \Sigma, \delta, q_0, F)\), where

1. \(Q\) is a finite set called the **states**,
2. \(\Sigma\) is a finite set called the **alphabet**,
3. \(\delta: Q \times \Sigma \rightarrow Q\) is the **transition function**,
4. \(q_0 \in Q\) is the **start state**, and
5. \(F \subseteq Q\) is the **set of accept states**.
Announcements

• HW 3 out
  • Due Mon 3/4 12pm EST (noon)

• HW 1 grades returned

• Use Gradescope re-grade request for all questions / complaints!
Is Concatenation Closed?

**THEOREM**

The class of regular languages is closed under the concatenation operation. In other words, if $A_1$ and $A_2$ are regular languages then so is $A_1 \circ A_2$.

*Proof requires:* Constructing **new** machine

- How does it know when to switch machines?
- Can only read input once
Let $M_1$ recognize $A_1$, and $M_2$ recognize $A_2$.

Want: Construction of $N$ to recognize $A_1 \circ A_2$.

\[ \varepsilon = \text{“empty transition”} = \text{reads no input} \]

Allows $N$ to be in both machines at the same time!

$N$ is an NFA! It can:
- Keep checking 1st part with $M_1$ and
- Move to $M_2$ to check 2nd part
Concatenation is Closed for Regular Langs

**Proof** (part of)

Let \( M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \) recognize \( A_1 \)

\( M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \) recognize \( A_2 \)

Construct \( N = (Q, \Sigma, \delta, q_1, F_2) \) to recognize \( A_1 \circ A_2 \)

1. \( Q = Q_1 \cup Q_2 \)
2. The state \( q_1 \) is the same as the start state of \( M_1 \)
3. The accept states \( F_2 \) are the same as the accept states of \( M_2 \)
4. Define \( \delta \) so that for any \( q \in Q \) and any \( a \in \Sigma, \)
Concatenation is Closed for Regular Langs

**Proof (part of)**

Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$
$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$

Construct $N = (Q, \Sigma, \delta, q_1, F_1)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state $q_1$ is the same as the start state of $M_1$
3. The accept states $F_2$ are the same as the accept states of $M_2$
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma$,

$$\delta(q, a) = \begin{cases} 
\{\delta_1(q, a)\} & q \in Q_1 \text{ and } q \notin F_1 \\
\{\delta_2(q, a)\} & q \in F_1 \text{ and } a \neq \epsilon \\
\{q_2\} & q \in F_1 \text{ and } a = \epsilon \\
\text{?} & q \notin Q_1 \cup F_1 
\end{cases}$$

And: $\delta(q, \epsilon) = \emptyset$, for $q \in Q, q \notin F_1$

Wait, is this true?
Is Concat Closed For Regular Langs?

Proof?

**Statements**
1. $A_1$ and $A_2$ are regular languages
2. A DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizes $A_1$
3. A DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizes $A_2$
4. Construct NFA $M = (Q, \Sigma, \delta, q_0, F)$
5. $M$ recognizes $A_1 \cup A_2$ and $A_1 \circ A_2$
6. $A_1 \cup A_2$ and $A_1 \circ A_2$ is a regular language
7. The class of regular languages is closed under concatenation operation.

**Justifications**
1. Assumption
2. Def of Reg Lang (Coro)
3. Def of Reg Lang (Coro)
4. Def of NFA
5. See Examples Table
6. ??? Does NFA recognize reg langs?
7. From stmt #1 and #6

In other words, if $A_1$ and $A_2$ are regular languages then so is $A_1 \circ A_2$. Q.E.D.
A DFA’s Language

• For DFA $M = (Q, \Sigma, \delta, q_0, F)$

• $M$ accepts $w$ if $\delta(q_0, w) \in F$

• $M$ recognizes language $\{w \mid M$ accepts $w\}$

Definition: A DFA’s language is a regular language
An NFA’s Language?

• For NFA $N = (Q, \Sigma, \delta, q_0, F)$
  
  $\text{Intersection ...}$
  $\text{... with accept states ...}$
  $\text{... is not empty set}$

• $N$ accepts $w$ if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$
  
  i.e., accept if final states contains at least one accept state

• Language of $N = L(N) = \{ w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset \}$

Q: What kind of languages do NFAs recognize?
Concatenation Closed for Reg Langs?

• Combining DFAs to recognize concatenation of languages ...
  ... produces an NFA

• So to prove concatenation is closed ...
  ... we must prove that NFAs also recognize regular languages.

Specifically, we must prove:
NFAs $\Leftrightarrow$ regular languages
“If and only if” Statements

\[ X \iff Y = \text{“} X \text{ if and only if } Y \text{”} = X \text{ iff } Y = X \iff Y \]

Represents two statements:

1. \( \Rightarrow \) if \( X \), then \( Y \)
   - “forward” direction

2. \( \Leftarrow \) if \( Y \), then \( X \)
   - “reverse” direction
How to Prove an “iff” Statement

\[ X \iff Y = \text{“}X \text{ if and only if } Y\text{”} = X \text{ iff } Y = X \iff Y \]

Proof has two (If-Then proof) parts:

1. \( \Rightarrow \) if \( X \), then \( Y \)
   - “forward” direction
   - assume \( X \), then use it to prove \( Y \)

2. \( \Leftarrow \) if \( Y \), then \( X \)
   - “reverse” direction
   - assume \( Y \), then use it to prove \( X \)
Proving NFAs Recognize Regular Langs

Theorem:
A language $L$ is regular if and only if some NFA $N$ recognizes $L$.

Proof: 2 parts

⇒ If $L$ is regular, then some NFA $N$ recognizes it.
   (Easier)
   • We know: if $L$ is regular, then a DFA exists that recognizes it.
   • So to prove this part: Convert that DFA → an equivalent NFA! (see HW 3)

⇐ If an NFA $N$ recognizes $L$, then $L$ is regular.

“equivalent” = “recognizes the same language”
⇒ If \( L \) is regular, then some NFA \( N \) recognizes it

<table>
<thead>
<tr>
<th>Statements</th>
<th>Justifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( L ) is a regular language</td>
<td>1. Assumption</td>
</tr>
<tr>
<td>2. A DFA ( M ) recognizes ( L )</td>
<td>2. Def of Regular lang (Coro)</td>
</tr>
<tr>
<td>3. Construct NFA ( N = \text{convert}(M) )</td>
<td>3. See hw 2 3!</td>
</tr>
<tr>
<td>4. DFA ( M ) is equivalent to NFA ( N )</td>
<td>4. See Equiv. table!</td>
</tr>
<tr>
<td>5. An NFA ( N ) recognizes ( L )</td>
<td>5. ???</td>
</tr>
<tr>
<td>6. If ( L ) is a regular language, then some NFA ( N ) recognizes it</td>
<td>6. ByStmts #1 and # 5</td>
</tr>
</tbody>
</table>

Assume the “if” part ...

... use it to prove “then” part
"Proving" Machine Equivalence (Table)

Let: DFA $M = (Q, \Sigma, \delta, q_0, F)$  
NFA $N = \text{convert}(M)$  
$\hat{\delta}(q_0, w) \in F$ for some string $w$

**If $M$ accepts $w$ ...**

Then we know ...

There is some sequence of states: $r_1 \ldots r_n$, where $r_i \in Q$ and $r_1 = q_0$ and $r_n \in F$

Then $N$ accepts?/rejects? $w$ because ...

Justification #1?  
There is an accepting sequence of set of states in $N$ ... for string $w$
“Proving” Machine Equivalence (Table)

Let: DFA $M = (Q, \Sigma, \delta, q_0, F)$

NFA $N = \text{convert}(M)$

$\hat{\delta}(q_0, w) \in F$ for some string $w$

$\hat{\delta}(q_0, w') \notin F$ for some string $w'$

<table>
<thead>
<tr>
<th>String</th>
<th>$M$ accepts?</th>
<th>$N$ accepts?</th>
<th>$N$ accepts? Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>Yes</td>
<td>???</td>
<td>See justification #1</td>
</tr>
<tr>
<td>$w'$</td>
<td>No</td>
<td>???</td>
<td>See justification #2?</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Proving NFAs Recognize Regular Langs

Theorem:
A language $L$ is regular if and only if some NFA $N$ recognizes $L$.

Proof:

$\Rightarrow$ If $L$ is regular, then some NFA $N$ recognizes it.
(Easier)
- We know: if $L$ is regular, then a DFA exists that recognizes it.
- So to prove this part: Convert that DFA → an equivalent NFA! (see HW 3)

$\Leftarrow$ If an NFA $N$ recognizes $L$, then $L$ is regular.
(Harder)
- We know: for $L$ to be regular, there must be a DFA recognizing it
- Proof Idea for this part: Convert given NFA $N$ → an equivalent DFA

“equivalent” = “recognizes the same language”
How to convert NFA→DFA?

A finite automaton is a 5-tuple \((Q, \Sigma, \delta, q_0, F)\), where

1. \(Q\) is a finite set called the states,
2. \(\Sigma\) is a finite set called the alphabet,
3. \(\delta: Q \times \Sigma \rightarrow Q\) is the transition function,
4. \(q_0 \in Q\) is the start state, and
5. \(F \subseteq Q\) is the set of accept states.

Proof idea:
Let each “state” of the DFA = set of states in the NFA

A nondeterministic finite automaton is a 5-tuple \((Q, \Sigma, \delta, q_0, F)\), where

1. \(Q\) is a finite set of states,
2. \(\Sigma\) is a finite alphabet,
3. \(\delta: Q \times \Sigma_\varepsilon \rightarrow \mathcal{P}(Q)\) is the transition function,
4. \(q_0 \in Q\) is the start state, and
5. \(F \subseteq Q\) is the set of accept states.
NFA computation can be in multiple states

DFA computation can only be in one state

So encode: a set of NFA states as one DFA state

This is similar to the proof strategy from “Closure of union” where: a state = a pair of states
Convert NFA→DFA, Formally

• Let NFA \( N = (Q, \Sigma, \delta, q_0, F) \)
• An equivalent DFA \( M \) has states \( Q' = \mathcal{P}(Q) \) (power set of \( Q \))
Example:

- Let $\text{NFA } N_4 = (Q, \Sigma, \delta, q_0, F)$
- An equivalent DFA $D$ has states $Q = \mathcal{P}(Q)$ (power set of $Q$)

The NFA $N_4$

A DFA $D$ that is equivalent to the NFA $N_4$
NFA→DFA

Have: NFA $N = (Q_{NFA}, \Sigma, \delta_{NFA}, q_{0NFA}, F_{NFA})$

Want: DFA $D = (Q_{DFA}, \Sigma, \delta_{DFA}, q_{0DFA}, F_{DFA})$

1. $Q_{DFA} = \mathcal{P}(Q_{NFA})$  
   A DFA state = a set of NFA states

2. For $qs \in Q_{DFA}$ and $a \in \Sigma$
   • $\delta_{DFA}(qs, a) = \bigcup_{q \in qs} \delta_{NFA}(q, a)$  
     A DFA step = an NFA step for all states in the set

3. $q_{0DFA} = \{q_{0NFA}\}$

4. $F_{DFA} = \{qs \in Q_{DFA} \mid qs \text{ contains accept state of } N\}$
Flashback: Adding Empty Transitions

- Define the set $\varepsilon$-REACHABLE($q$)
  - ... to be all states reachable from $q$ via zero or more empty transitions

(Defined recursively)

- **Base case:** $q \in \varepsilon$-REACHABLE($q$)

- **Recursive case:**
  $$\varepsilon$\text{-REACHABLE}(q) = \{ r \mid p \in \varepsilon$\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon) \}$$

A state is in the reachable set if ...

... there is an empty transition to it from another state in the reachable set
NFA → DFA

Have: NFA $N = (Q_{NFA}, \Sigma, \delta_{NFA}, q_{0NFA}, F_{NFA})$

Want: DFA $D = (Q_{DFA}, \Sigma, \delta_{DFA}, q_{0DFA}, F_{DFA})$

1. $Q_{DFA} = \mathcal{P}(Q_{NFA})$

2. For $q_s \in Q_{DFA}$ and $a \in \Sigma$
   - $\delta_{DFA}(q_s, a) = \bigcup_{q \in \operatorname{\varepsilon\text{-REACHABLE}}(q)} \delta_{NFA}(q, a)$

3. $q_{0DFA} = \{q_{0NFA}\} \cup \operatorname{\varepsilon\text{-REACHABLE}}(q_{0NFA})$

4. $F_{DFA} = \{q_s \in Q_{DFA} \mid q_s \text{ contains accept state of } N\}$
Proving NFAs Recognize Regular Langs

Theorem: A language $L$ is regular if and only if some NFA $N$ recognizes $L$.

Proof:
$\Rightarrow$ If $L$ is regular, then some NFA $N$ recognizes it.
   (Easier)
   • We know: if $L$ is regular, then a DFA exists that recognizes it.
   • So to prove this part: Convert that DFA $\rightarrow$ an equivalent NFA! (see HW 3)

$\Leftarrow$ If an NFA $N$ recognizes $L$, then $L$ is regular.
   (Harder)
   • We know: for $L$ to be regular, there must be a DFA recognizing it
   • Proof Idea for this part: Convert given NFA $N$ $\rightarrow$ an equivalent DFA ...
     $\ldots$ using our NFA to DFA algorithm!
Concatenation is Closed for Regular Langs.

**Proof**

Let \( M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \) recognize \( A_1 \)
\( M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \) recognize \( A_2 \)

Construct \( N = (Q, \Sigma, \delta, q_1, F) \) to recognize \( A_1 \circ A_2 \)

1. \( Q = Q_1 \cup Q_2 \)
2. The state \( q_1 \) is the same as the start state of \( M_1 \)
3. The accept states \( F_2 \) are the same as the accept states of \( M_2 \)
4. Define \( \delta \) so that for any \( q \in Q \) and any \( a \in \Sigma_\epsilon \),

\[
\delta(q, a) = \begin{cases} 
\{\delta_1(q, a)\} & q \in Q_1 \text{ and } q \not\in F_1 \\
\{\delta_1(q, a)\} & q \in F_1 \text{ and } a \neq \epsilon \\
\{q_2\} & q \in F_1 \text{ and } a = \epsilon \\
\{\delta_2(q, a)\} & q \in Q_2. \quad \text{And: } \delta(q, \epsilon) = \emptyset, \text{ for } q \in Q, q \not\in F_1 \end{cases}
\]
New possible proof strategy!

Concat Closed for Reg Langs: Use NFAs Only

**Proof**

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$, and

$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$.

Construct $N = (Q, \Sigma, \delta, q_1, F_1)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$
2. The state $q_1$ is the same as the start state of $N_1$
3. The accept states $F_1$ are the same as the accept states of $N_2$
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma$, 

$$
\delta(q, a) = \begin{cases}
\delta_1(q, a) & q \in Q_1 \text{ and } q \not\in F_1 \\
\delta_2(q, a) & q \in Q_2 \\
q_1 & q \in F_1 \text{ and } a = \epsilon \\
\delta_1(q, a) & q \in F_1 \text{ and } a \neq \epsilon
\end{cases}
$$

If language is regular, then it has an NFA recognizing it...
Flashback: Union is Closed For Regular Langs

**Theorem**

The class of regular languages is closed under the union operation. In other words, if $A_1$ and $A_2$ are regular languages, so is $A_1 \cup A_2$.

**Proof:**

- How do we prove that a language is regular?
  - Create a DFA or NFA recognizing it!
- Combine the machines recognizing $A_1$ and $A_2$
  - Should we create a DFA or NFA?
Flashback: Union is Closed For Regular Langs

Proof

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize $A_1$, $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize $A_2$,

- Construct: a new machine $M = (Q, \Sigma, \delta, q_0, F)$ using $M_1$ and $M_2$

- states of $M$: $Q = \{(r_1, r_2) | r_1 \in Q_1 \text{ and } r_2 \in Q_2\} = Q_1 \times Q_2$
  This set is the Cartesian product of sets $Q_1$ and $Q_2$

- $M$ transition fn: $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$

- $M$ start state: $(q_1, q_2)$

- $M$ accept states: $F = \{(r_1, r_2) | r_1 \in F_1 \text{ or } r_2 \in F_2\}$
Union is Closed for Regular Languages

Add new start state, and $\varepsilon$-transitions to old start states
Union is Closed for Regular Languages

**PROOF**

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$, and

$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state $q_0$ is the start state of $N$.
3. The set of accept states $F = F_1 \cup F_2$. 
Union is Closed for Regular Languages

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$, and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state $q_0$ is the start state of $N$.
3. The set of accept states $F = F_1 \cup F_2$.
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma$, 

\[
\delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \\
\delta_2(q, a) & q \in Q_2 \\
\{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\
\emptyset & q = q_0 \text{ and } a \neq \varepsilon 
\end{cases}
\]

Don’t forget Statements and Justifications!
List of Closed Ops for Reg Langs (so far)

- Union
- Concatentation

  - Kleene Star (repetition)?
Kleene Star Example

Let the alphabet $\Sigma$ be the standard 26 letters $\{a, b, \ldots, z\}$. If $A = \{\text{good, bad}\}$

$$A^* = \{\varepsilon, \text{good, bad, goodgood, goodbad, badgood, badbad, goodgoodgood, goodgoodbad, goodbadgood, goodbadbad, \ldots} \}$$

Note: repeat zero or more times

(this is an infinite language!)
New start (and accept) state, $\epsilon$-transitions to old start state

Old accept states $\epsilon$-transition to old start state
Kleene Star is Closed for Regular Langs

**THEOREM**

The class of regular languages is closed under the star operation.
Kleene Star is Closed for Regular Langs

(part of)

**Proof**

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$.

Construc $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1^*$. 

1. $Q = \{q_0\} \cup Q_1$

2. The state $q_0$ is the new start state.

3. $F = \{q_0\} \cup F_1$

Kleene star of a language must accept the empty string!
Kleene Star is Closed for Regular Langs

(part of)

**Proof** Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1^\ast$.

1. $Q = \{q_0\} \cup Q_1$

2. The state $q_0$ is the new start state.

3. $F = \{q_0\} \cup F_1$

4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_e$,

$$\delta(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \text{ and } q \not\in F_1 \\
\delta_1(q, a) & q \in F_1 \text{ and } a \neq \epsilon \\
\delta_1(q, a) \cup \{q_1\} & q \in F_1 \text{ and } a = \epsilon \\
\{q_1\} & q = q_0 \text{ and } a = \epsilon \\
\emptyset & q = q_0 \text{ and } a \neq \epsilon.
\end{cases}$$
Next Time: Why These Closed Operations?

• Union
• Concat
• Kleene star

All regular languages can be **constructed** from:
- single-char strings, and
- these three **combining** operations!
Submit in-class work 2/26

On gradescope