Proving Languages Non-Regular

UMB CS 622

Wednesday March 9, 2024
Announcements

• HW 4 out
  • Due Mon 3/18 12pm EST (noon)
  • (After spring break)

• Problem 4, Part 2c Update:
  • Prove the statement for
    • 1 base case
    • 1 recursive case

• No class next week! (Spring Break)
Prove: Spider-Man does not exist  ???

We know how to: prove a language is regular
Can we: prove a language is not regular?

Proving something not true is different (and usually harder) than proving it true

It’s sometimes possible, but often needs new proof techniques!
Quantified Logical Statements

• “Exists” (Existential)
  • “Easier” to prove TRUE
    • Just need **one** example!

  \[ \exists x P(x) \] is true when \( P(x) \) is true for at least **one** value of \( x \).

  “There exists a natural number \( n \) such that, \( n \cdot n = 25 \)”

• “For all” (Universal)
  • “Harder” to prove TRUE
    • Need to prove true for **all** examples

  \[ \forall x P(x) \] is true when \( P(x) \) is true for **all** values of \( x \).

  “For all natural numbers \( n \), \( 2 \cdot n = n + n \)”
Quantified Logical Statements in CS 622

• “Exists” (Existential)
  • “Easier” to prove TRUE
    • Just need one example!
  • “Harder” to prove FALSE
    • Need to prove false for all examples

• “For all” (Universal)
  • “Harder” to prove TRUE
    • Need to prove true for all examples
  • “Easier” to prove FALSE
    • Just need one (counter)example!

Language $L$ is regular

Language $L$ is not regular?

There exists one DFA that recognizes $L$

There are no possible DFAs that recognizes $L$

For all regular languages $L$, $L^*$ is a regular language

For all strings in a regular language ...

Key is finding such a statement about regular languages!
**Pumping lemma** If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^i z \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

For all (long enough) strings in the language ...!

... they have this **repeatable structure** (Kleene star)

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**Why is this true?**

Because if you give DFA an input > # states, then some state repeats! i.e., “long enough strings” start to show repeating pattern!
Equivalence of Conditional Statements

• Yes or No? “If $X$ then $Y$” is equivalent to:
  
  • “If $Y$ then $X$” (converse)
    • No!
  
  • “If not $X$ then not $Y$” (inverse)
    • No!
  
  • “If not $Y$ then not $X$” (contrapositive)
    • Yes!
Pumping lemma  If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Equivalent (contra-positive): If the “for all” is not true ... 
le if just one counterexample string doesn’t have the repeatable structure ...

Contrapositive: “If $X$ then $Y$” is equivalent to “If not $Y$ then not $X$”
Kinds of Mathematical Proof

• Deductive Proof
  • Logically infer conclusion from known definitions and assumptions

• Proof by induction
  • Use to prove properties of recursive definitions or functions

• Proof by contradiction
  • Proving the contrapositive
How To Do Proof By Contradiction

3 easy steps:
1. **Assume:** the opposite of the statement to prove
2. **Show:** the assumption leads to a contradiction
3. **Conclude:** the original statement must be true
Pumping Lemma: Non-Regularity Example

This repetition pattern cannot be expressed with a star regular expression?

Let $B$ be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that $B$ is not regular. The proof is by contradiction.
Want to prove: \(0^n1^n\) is not a regular language

Proof (by contradiction):

- **Assume:** \(0^n1^n\) is a regular language
  - So it must satisfy the pumping lemma
  - i.e., all strings \(\geq\) length \(p\) are pumpable
- **Counterexample** = \(0^p1^p\)
  - **In the language**
  - **Greater than length \(p\)**
  - **Should be able to split into \(xyz\) where \(y\) is pumpable**

We must show that there is no possible way to split this string to satisfy the conditions of the pumping lemma!

Reminder: Pumping lemma says: all strings \(0^n1^n \geq\) length \(p\) are splittable into \(xyz\) where \(y\) is pumpable

So find counterexample string \(\geq\) length \(p\) that is not splittable into \(xyz\) where \(y\) is pumpable
**Want to prove:** $0^n1^n$ is not a regular language

**Possible Split:** $y =$ all 0s

**Proof (by contradiction):**

- **Assume:** $0^n1^n$ is a regular language
  - So it must satisfy the pumping lemma
  - i.e., all strings ≥ length $p$ are pumpable
- **Counterexample** = $0^p1^p$
- **Choose** $xyz$ split so $y$ contains:
  - all 0s
- **Pumping $y$: produces a string with more 0s than 1s**
  - ... which is not in the language $0^n1^n$!
  - If $0^p1^p$ is not pumpable? (according to pumping lemma)
  - Then $0^n1^n$ is a not regular language? (contrapositive)
  - This is a contradiction of the assumption?

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**Pumping Lemma:** If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

**Contrapositive:** If not true ...

- So find counterexample string ≥ length $p$ that is not splittable into $xyz$ where $y$ is pumpable
- $p$ 0s
- $p$ 1s
- 00 ... 011 ... 1
- $x$ $y$ ??? $z$
- BUT ... pumping lemma requires only one pumpable splitting
- This one didn’t work, but proof is not done!
- Is there another way to split into $xyz$?
Want to prove: $0^n1^n$ is not a regular language

Possible Split: \( y = \text{all } 1\text{s} \)

Proof (by contradiction):

- **Assume:** $0^n1^n$ is a regular language
  - So it must satisfy the pumping lemma
  - I.e., all strings $\geq$ length \( p \) are pumpable

- **Counterexample** = \( 0^p1^p \)

- Choose \( xyz \) split so \( y \) contains:
  - all 1s

\[
\begin{array}{c}
\text{Is there another way} \\
\text{to split into } xyz \text{?}
\end{array}
\]

- Is this string pumpable (repeating \( y \) produces string still in $0^n1^n$)?
  - No!
  - By the same reasoning as in the previous slide
Want to prove: $0^n 1^n$ is not a regular language

Possible Split: $y = 0s$ and $1s$

Proof (by contradiction):

• Assume: $0^n 1^n$ is a regular language
  • So it must satisfy the pumping lemma
  • I.e., all strings $\geq$ length $p$ are pumpable

• Counterexample = $0^p 1^p$

• Choose $xyz$ split so $y$ contains:
  • both $0s$ and $1s$

  $$x y z$$

  $$00 \ldots 011 \ldots 1$$

  $$p 0s \quad p 1s$$

• Is this string pumpable (repeating $y$ produces string still in $0^n 1^n$)?
  • No!
  • Pumped string will have equal $0s$ and $1s$ …
  • But they will be in the wrong order: so there is still a contradiction!

Yes! QED

But maybe we didn’t have to …

Did we examine every possible splitting?
The Pumping Lemma: Condition 3

**Pumping lemma**  If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

$p$ 0s  

didn’t have to look at other possible splittings

00 ... 011 ... 1

The repeating part $y$ ... must be in the first $p$ characters!

i.e., “long enough strings” start to show repeating pattern!

$y$ must be in here!
The Pumping Lemma: Pumping Down

**Pumping lemma** If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Repeating party must be non-empty ... but can be repeated zero times!

Example: $L = \{0^i1^j \mid i > j\}$
Want to prove: $L = \{0^i1^j \mid i > j\}$ is not a regular language

**Proof (by contradiction):**

- **Assume: $L$ is a regular language**
  - So it must satisfy the pumping lemma
  - I.e., all strings $\geq$ length $p$ are pumpable
- **Counterexample** = $0^{p+1}1^p$
- Choose $xyz$ split so $y$ contains:
  - all 0s
  - (Only possibility, by condition 3)
- Repeat $y$ zero times (pump down): produces string with $\#$ 0s $\leq$ $\#$ 1s
  - ... not in the language $\{0^i1^j \mid i > j\}$
  - So $\{0^i1^j \mid i > j\}$ does not satisfy the pumping lemma
  - So it is a not regular language
  - This is a contradiction of the assumption!
Next Time (and rest of the Semester)

• If a language is not regular, then what is it?

• There are many more classes of languages!