Data Abstractions and Pairs

September 21, 2020
What to read in Chapter 1

We have already covered sections 1.1.1–1.2.4 and section 1.3.2 (lambda).
You should also read

- 1.3.1 Procedures as arguments. (Read this carefully.)
- 1.3.2 Lambda (We already did this.)
- 1.3.3 Processes as general methods.
- 1.3.4 Procedures as returned values.

These last two sections you can read quickly and not so carefully.
The \textit{let} special form

Suppose we want to evaluate an expression like this:

\[
f(x, y) = (x + y + xy)^2 + (x + y - xy)^2
\]

It would be nice to be able to compute \(x + y\) and \(xy\) only once, rather than twice. \((x + y\) and \(xy\) are what compiler writers call \textit{common subexpressions}.\) We can do this like this:

\begin{verbatim}
(define (f x y)
  (let ((a (+ x y))
        (b (* x y)))
    (+ (square (+ a b))
       (square (- a b)))))
\end{verbatim}
Actually, the `let` special form is really “syntactic sugar” – it is transformed internally as follows:

```
(let ((var1 exp1) ((lambda (var1 var2 ... varn)
   (var2 exp2) ==> body)
   ...
   (varn expn))
   body)
```

```
((lambda (var1 var2 ... varn)
   (var2 exp2) ==> body)
   ...
   (varn expn))
```

```
exp1 exp2 ... expn)
```
So far we’ve seen primitive expressions and functions.

We used abstractions to build higher order procedures.

Sometimes we want to “glue together” several data objects.

We can use abstraction to build compound data as well.

For example, suppose Scheme did not contain support for fractions (i.e., for rational numbers represented as fractions).

A rational number is essentially a pair of numbers – a numerator and a denominator.

We could represent them separately but it would be complicated and confusing, especially if we want to define arithmetic operations on rational numbers.

We’d need to keep track on which numerators and denominators belong where...

We want to be able to “glue together” the two numbers.
We could create a rational number package by writing procedures of the following form:

- A *constructor*
  
  (make-rat n d)

- Two *selectors*
  
  (numer x)
  
  (denom x)
Example – Rational Numbers

These procedures must be related in the following ways:

\[
\begin{align*}
\text{(numer (make-rat n d))} & \implies n \\
\text{(denom (make-rat n d))} & \implies d \\
\text{(make-rat (numer x) (denom x))} & \implies x
\end{align*}
\]

Assuming we have these procedures, we can construct procedures to implement the usual operators. For instance, we know that

\[
\frac{n_1}{d_1} + \frac{n_2}{d_2} = \frac{n_1 d_2 + n_2 d_1}{d_1 d_2}
\]

with the other usual formulas for the other arithmetic operations.
(define (add-rat x y)
  (make-rat (+ (* (numer x)(denom y))
              (* (numer y)(denom x)))
              (* (denom x)(denom y)) ))

(define (sub-rat x y)
  (make-rat (- (* (numer x)(denom y))
              (* (numer y)(denom x)))
              (* (denom x)(denom y)) ))

(define (mul-rat x y)
  (make-rat (* (numer x)(numer y))
              (* (denom x)(denom y)) ))

(define (div-rat x y)
  (make-rat (* (numer x)(denom y))
              (* (denom x)(numer y)) ))
and also

\[(\text{define (equal-rat? x y)}\]
\[\text{ (= (* (numer x)(denom y))}\]
\[\text{ (* (denom x)(numer y)))})\]

\[(\text{define (print-rat x)}\]
\[\text{(display* (numer x) "/" (denom x)))} \]

So everything works out fine, as long as we can implement the constructor and the selectors.
The basic constructors and selectors for data structures in Scheme are these:

- **constructor**: cons
- **selectors**: car and cdr (read: could-er)

```
(define x (cons 1 2))
```

We say that `cons` creates a *pair*. With this definition,

- `(car x) ==> 1`
- `(cdr x) ==> 2`

Suppose we also define

```
(define y (cons 3 4))
(define z (cons x y))
```
Pairs

(car (car z)) ==> 1 ;; this is also produced by (caar z)
(car (cdr z)) ==> 3 ;; this is also produced by (cadr z)

So now we can implement our rational number package as follows:

(define (make-rat n d)
  (cons n d))

(define (numer x)
  (car x))

(define (denom x)
  (cdr x))

(Incidentally, we could just as well have written

(define make-rat cons)
Of course we have not dealt at all with questions of efficiency, representations, and lowest terms.

For instance, should rational numbers be stored internally in lowest terms, or should they only be put into lowest terms when they are output?

There are tradeoffs to be made here.
But in any case, we have the following hierarchy of objects and procedures:

1. What the user sees is rational numbers, together with arithmetic operations on them (add-rat, etc.).

2. Those arithmetic operations are implemented in terms of the constructors and selectors (make-rat, etc.) and in terms of Scheme’s primitive arithmetic operations on integers (+, etc.).

3. The constructors and selectors are implemented in terms of Scheme’s primitive constructors and selectors (cons, car, and cdr).

4. cons, car, and cdr are implemented in some fashion. In fact, we have said nothing about how pairs are implemented in Scheme.
What really is a pair? It could be implemented internally in various ways.

It could be any data structure so long as cons, car, and cdr work together properly.

In fact, it might not be a data structure at all, but a function:

```
(define (cons x y)
  (define (dispatch m)
    (cond ((= m 0) x)
          ((= m 1) y)
          (else (error "Improper argument")))
  dispatch)

(define (car z) (z 0))
(define (cdr z) (z 1))
```
Another Way to Define cons

(define (cons x y)
  (lambda (m)
    (cond ((= m 0) x)
          ((= m 1) y)
          (else (error "Improper argument"))))

Thus in Scheme, the difference between data and procedures fades away. This will be very important in this course.
The conventional way to represent a pair (say, (cons 1 2)) is like this:
The empty list is represented in a Scheme expression as ’(). (We’ll talk about that single quote mark later.)

The list consisting of the numbers 1, 2, 3, 4 in that order is created by the constructor (list 1 2 3 4).

This is syntactic sugar for the following:

(cons 1
  (cons 2
    (cons 3
      (cons 4
        ’() ))))
Lists (or sequences)

That is, it is turned internally into the following data structure:

```
1
2
3
4
()
```

or

```
1 2 3 4
```

```
[ ] [ ] [ ] [ ]
1 2 3 4
```

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Lists (or sequences)

- So a list is actually constructed internally out of pairs.
- That is, it’s really a special kind of binary tree.
- On output, pairs are represented with dots. For instance:

  => (cons 1 2)

  (1 . 2)
  =>

  However, when the data structure is actually a list, it is represented as such.
Lists (or sequences)

The above list could be represented on output as

\[(1 \cdot (2 \cdot (3 \cdot (4 \cdot ()))))\]

But that's not what actually happens:

\[\Rightarrow (\text{list } 1 2 3 4)\]

\[1 2 3 4\]
\[\Rightarrow (\text{cons } 1\]

\[1 2 3 4\]
\[\Rightarrow (\text{cons } 2\]

\[1 2 3 4\]
\[\Rightarrow (\text{cons } 3\]

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\[\Rightarrow (\text{cons } 4\]

\[1 2 3 4\]
Display vs. Representation

Internal Data Structure  Output Representation  Input Representation

1
   2
      3
         4

(1 2 3 . 4)  
(cons 1 (cons 2 (cons 3 4)))

1
   2
      3
         4

(1 2 (3 . 4))  
(list 1 2 (cons 3 4))

1
   2
      3
         4

(1 (2 3) 4)  
(list 1 (list 2 3) 4)
<table>
<thead>
<tr>
<th>Internal Data Structure</th>
<th>Output Representation</th>
<th>Input Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="1" alt="Diagram" /></td>
<td>(1 2 3 . 4)</td>
<td>(cons 1 (cons 2 (cons 3 4)))</td>
</tr>
<tr>
<td><img src="2" alt="Diagram" /></td>
<td>(1 2 (3 . 4))</td>
<td>(list 1 2 (cons 3 4))</td>
</tr>
<tr>
<td><img src="3" alt="Diagram" /></td>
<td>(1 (2 3) 4)</td>
<td>(list 1 (list 2 3) 4)</td>
</tr>
</tbody>
</table>
Note that if

\[(\text{define } x \ (\text{list} \ 1 \ 2 \ 3 \ 4))\]

Then

\[\ (\text{car} \ x) \ \text{is} \ 1 \ \text{an element of the list} \]
\[\ (\text{cdr} \ x) \ \text{is} \ (2 \ 3 \ 4) \ \text{the rest of the list} \]

and further,

\[\ (\text{cons} \ 10 \ x) \ \text{is} \ (10 \ 1 \ 2 \ 3 \ 4) \]
\[\ (\text{caddr} \ x) \ \text{is} \ 3 \]
\[\ (\text{cddddr} \ x) \ \text{is} \ () \]
(define (count-down-from n)
  (cond ((= n 0) (display 0) (newline))
    (else (display n) (newline)
      (count-down-from (- n 1)))))

**Question:** Why can’t we just use “if” here?

(define (count-up-to n)
  (cond ((= n 0) (display 0) (newline))
    (else (count-up-to (- n 1))
      (display n) (newline))))

(define (add-elements items)
  (if (null? items)
    0
    (+ (car items) (add-elements (cdr items)))))
(define (add-squares items)
    (if (null? items)
        0
        (+ (square (car items))
            (add-squares (cdr items)))))

(define (add-transformed items func)
    (if (null? items)
        0
        (+ (func (car items))
            (add-transformed (cdr items) func))))

(define (list-ref items n)
    (if (= n 0)
        (car items)
        (list-ref (cdr items) (- n 1)))))
Procedures that manipulate lists

(define (length items)
  (if (null? items)
      0
      (+ 1 (length (cdr items))))
)

Here is an iterative form of length:

(define (length items)
  (length-iter items 0))

(define (length-iter items result)
  (if (null? items)
      result
      (length-iter (cdr items) (+ result 1))
  )
)
(define (append x y)
  (if (null? x)
      y
      (cons (car x) (append (cdr x) y)) ))

==> (define squares (list 1 4 9 16 25))

squares
==> (define odds (list 1 3 5 7 9))

odds
==> (append squares odds)

(1 4 9 16 25 1 3 5 7 9)
==>
Procedures that manipulate lists

- See a pattern here? We have a function f, we apply it to the car and then recursively to the cdr.
- Most examples we saw behave in a similar way, so we can abstract even more...
- Write a higher order procedure that takes a function and applies it to a sequence.

```
(define (map func sequence)
  (if (null? sequence)
      '()
      (cons (func (car sequence))
            (map func (cdr sequence))))
```

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Procedures that manipulate lists

```scheme
==> (map abs (list -10 2.5 -11.6 17))

(10 2.5 11.6 17)

==> (map (lambda (x) (* x x))
       (list 1 2 3 4))

(1 4 9 16)

==> 

(define (filter predicate sequence)
    (cond ((null? sequence) '())
          ((predicate (car sequence))
             (cons (car sequence)
                   (filter predicate (cdr sequence))))
          (else (filter predicate (cdr sequence)))))
```