1. (15%) **Amortized analysis:** Suppose we perform a sequence of \( n \) operations on a data structure in which the \( i^{th} \) operation costs \( i \) if \( i \) is an exact power of 2, and 1 otherwise. Show that the amortized cost per operation is \( O(1) \). Use any method we showed in class.

**Answer:** This is very similar to the dynamic (Hash) table example shown in class. We showed that the amortized cost per operation is 3, but here is a recap: Say we charge $3 for every operation. if the operation is not a power of 2, we use $1 and save $2. When we reach a power of 2 \( n = 2^i \) for some \( i \), the amount of money we saved is equal to the number of operations since the last power of 2, which is \( \sum_{2^{i-1}+1}^{2^i-1} 2 = 2 \cdot (2^{i-1} - 1) = 2^i - 2 \). If we add the $3 we pay for the \( n^{th} \) operation itself, we have enough to cover it.

2. (20%) **Graphs:** In this question we refer to a node as “undiscovered” if it has not been seen yet by a graph walk (colored white in class). A node is “discovered” if it has been seen but not yet done (colored gray in class). A node is “processed” if it is done (colored black in class).

   (a) (10%) Show an example of a graph \( G = (V, E) \) such that a BFS walk yields \( O(|V|) \) processed (gray) nodes at some given moment. Explain. **To get the full mark your example should contain at least four nodes.**

   **Answer:** See figure:

```
  a
 /|
/ |
 b a
```

When the BFS starts in \( a \). When \( a \) is done, all the other vertices are gray.

   (b) (10%) Show an example of a graph \( G = (V, E) \) such that a DFS walk yields \( O(|V|) \) processed (gray) nodes at some given moment. Explain. **To get the full mark your example should contain at least four nodes.**

   **Answer:** See figure:

```
  a  b
    |
    c
    d
```

When the BFS starts in \( a \). When \( d \) is explored, all vertices are gray.

3. (20%) **Flow:** Given the following flow graph:
Show the maximum flow for this graph. No need to show the residual graphs. Write the augmenting paths you find, the value of the flow for each path and the value of the overall maximum flow.

**Answer:**

The paths are:

- $s \to a \to b \to t$ (5)
- $s \to c \to d \to t$ (5)
- $s \to c \to b \to t$ (5)

The overall flow is 15.

4. (25%) **NP:** The directed Hamiltonian Path (HAM-Path) is the problem of finding a simple path that goes through every vertex in a directed graph once. To show that the problem is NP-complete we use a reduction from the directed Hamiltonian cycle problem (HAM-Cycle) – we start from an instance $G = (V, E)$ to the HAM-cycle problem. We create a directed graph $H = (V', E')$ as follows: Select an arbitrary node $u \in V$ and split it to two nodes, $u_{in}$ and $u_{out}$. Every edge $(u, v)$ in $G$ will now become $(u_{out}, v)$ in $H$ and every edge $(v, u)$ becomes $(v, u_{in})$ in $H$. The other vertices and edges remain unchanged.

(a) (6%) Show that directed HAM-Path is in NP.

**Answer:** This one is easy. Given the vertices in order, all we have to do is verify they constitute a simple path (no cycles) and that they are all the vertices in the graph. This takes linear time in the number of vertices.

(b) (7%) Show that the reduction described above is polynomial.

**Answer:** All we have to do is to add at most $|V| - 1$ edges from $u_{out}$ to all the rest, and at most $|V| - 1$ edges from all the rest to $u_{in}$ (depending on the degree of $u$). This is linear in the number of vertices, and we add a linear number of edges and one vertex, so the space is polynomial.

(c) (7%) Show that the graph $G$ has a HAM-Cycle iff $H$ has a HAM-Path. Don’t forget the two directions.

**Answer:**

⇒ If $G$ has a HAM-Cycle $\{v_1 = u, v_2, \ldots, v_n, v_1 = u\}$ (since it’s a cycle it doesn’t matter where we start the cycle, so we start with $u$). This corresponds to the path $\{u_{out}, v_2, \ldots, v_n, u_{in}\}$. Since there is an edge $(u, v_2)$ in the original graph, there is an edge $(u_{out}, v_2)$ in $H$, and since there is an edge $(v_n, u)$ in the original graph, there is an edge $(v_n, u_{out})$ in $H$.

⇐ If $H$ has a HAM-PATH, it must start with $u_{out}$ and end with $u_{in}$, since they don’t have incoming and outgoing edges, respectively. But they both represent the same node $u$ in $G$, so this path corresponds to a cycle in $G$.

5. (20%) If a graph $G$ is a directed acyclic graph (DAG), the HAM-Path problem can be solved in polynomial time.

(a) (10%) Describe a polynomial time algorithm to find whether $G$ has a HAM-path. Obviously, you will have to topologically sort the graph on the way. **Hint:** There is very little left to do after the topological sorting... But you have to explain carefully what should be done next.

**Answer:** First, topologically sort the graph. Second, check whether the topological order constitutes a path. If so, it is a HAM-Path, since a topological sort contains all the vertices.
(b) (10%) Show that if a DAG has a HAM-Path, it only has one possible topological sort.

**Answer:** According to the definition of a topological sort, if there is a path from $u$ to $v$ in a DAG, $u$ must appear before $v$ in any topological sort. If there were more than one topological sorts, it means there are at least two vertices with no path between them. This means there can't be a HAM-Path in the graph.