1. Order of growth:

Use the substitution method to show that $T(n) = 8T(n/2) + n^3 = O(n^3 \log n)$. Assume $T(2) = d$ (some constant). I know this can be shown easily with the master theorem, but please use induction and do not skip stages.

**Answer:**

(a) Base case: $T(2) = d \leq c * 2^3 \log 2 = 8c$ for the right choice of $c$.

(b) Inductive hypothesis: Assume $T(k) \leq ck^3 \log k$ for any $k < n$.

(c) Substitute in the equation to prove for $n$:

$$T(n) \leq 8(c(n/2)^3 \log n/2) + n^3 = 8c(n^3/8 \log n - \log 2) + n^3 = cn^3 \log n - (c - 1)n^3 \leq cn^3 \log n$$

for every $c \geq 1$.

2. Sorting

(a) Let $S$ be an unsorted array of $n$ integers. Give an algorithm that finds the pair $x, y \in S$ that maximizes $|x - y|$. Your algorithm must run in $O(n)$ worst-case time.

**answer:** The pair that maximizes $|x - y|$ must be the smallest and largest, since this is the maximum difference between any two numbers in the set. We saw in class how to find minimum and maximum (separately or simultaneously).

As we saw in class, finding the minimum and maximum is linear. Therefore the runtime is $O(N)$.

(b) Let $S$ be an unsorted array of $n$ integers. Give an algorithm that finds the pair $x, y \in S$ that minimizes $|x - y|$. Your algorithm must run in $O(n \log n)$ worst-case time.

**answer:** You sort the array using any $O(n \log n)$ algorithm, and the pair that minimizes $|x - y|$ must be a consecutive pair in the array. It is the pair with the smallest difference. So we go over the sorted array, finding the consecutive pair with the smallest difference. This adds $O(n)$, so the overall runtime is still $O(n \log n)$.

3. Heaps: In a max-heap of size $n$, represented as discussed in class, in what index(es) can the smallest element reside? Explain carefully. Assume all the $n$ numbers are different.

**answer:** The smallest element must be a leaf, since any non-leaf has at least one child and according to the definition of a max-heap, a parent must be larger than its child(ren). As we saw in class, the leaves are in indices $\lfloor n/2 \rfloor$...$n$. 