Greedy Algorithms

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Greedy Algorithms

- Like dynamic programming, used to solve optimization problems.
- Problems exhibit optimal substructure (like DP).
- Problems also exhibit the greedy-choice property.
- When we have a choice to make, make the one that looks best right now.
- Make a locally optimal choice in hope of getting a globally optimal solution.
The choice that seems best at the moment is the one we go with.

Prove that when there is a choice to make, one of the optimal choices is the greedy choice.

Therefore, it's always safe to make the greedy choice.

Show that all but one of the subproblems resulting from the greedy choice are empty.
A way to compress a text message.

Example: 100,000 characters, with only the letters \{a, b, c, d, e, f\}.

Fixed length coding:

<table>
<thead>
<tr>
<th>character</th>
<th>code</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>000</td>
</tr>
<tr>
<td>b</td>
<td>001</td>
</tr>
<tr>
<td>c</td>
<td>010</td>
</tr>
<tr>
<td>d</td>
<td>011</td>
</tr>
<tr>
<td>e</td>
<td>100</td>
</tr>
<tr>
<td>f</td>
<td>101</td>
</tr>
</tbody>
</table>

We need three bits for each character, so the entire message will take 300,000 bits to encode. Can we do better?
Using codes of variable lengths to encode characters.

The length is proportional to the frequency of the character.

Suppose the frequencies of the characters are as follows:

- We could do better if $a$ had a shorter code than $f$.

<table>
<thead>
<tr>
<th>character</th>
<th>times used</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>45,000</td>
</tr>
<tr>
<td>$b$</td>
<td>13,000</td>
</tr>
<tr>
<td>$c$</td>
<td>12,000</td>
</tr>
<tr>
<td>$d$</td>
<td>16,000</td>
</tr>
<tr>
<td>$e$</td>
<td>9,000</td>
</tr>
<tr>
<td>$f$</td>
<td>5,000</td>
</tr>
</tbody>
</table>
Prefix Codes

- A set of codes such that no code is the prefix of another
- This is the only way we know when one code ends and another one begins. For example:

<table>
<thead>
<tr>
<th>character</th>
<th>Frequency</th>
<th>code</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>.45</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>.13</td>
<td>101</td>
</tr>
<tr>
<td>c</td>
<td>.12</td>
<td>100</td>
</tr>
<tr>
<td>d</td>
<td>.16</td>
<td>111</td>
</tr>
<tr>
<td>e</td>
<td>.9</td>
<td>1101</td>
</tr>
<tr>
<td>f</td>
<td>.5</td>
<td>1100</td>
</tr>
</tbody>
</table>

The total size of the encoded message is now

\[(45 \cdot 1 + 13 \cdot 3 + 12 \cdot 3 + 16 \cdot 3 + 9 \cdot 4 + 5 \cdot 4) \cdot 1000 = 224,000 \text{ bits}\]

which is a significant improvement, even though some of the code words are actually longer this time.
If we treat the frequency as the relative number of times a character appears in the code, then we can re-write the former equation as:

\[ 1(0.45) + 3(0.13) + 3(0.12) + 3(0.16) + 4(0.09) + 4(0.05) = 2.24 \]

This is the expected number (or “average” number) of bits per character – as opposed to 3 bits per character in our fixed-length encoding.
Prefix Codes

- We can measure the efficiency of a code by the expected number of bits per character.
- Let $C$ be the set of characters.
- $x$ is a variable that runs over the set of characters in $C$, and if $f(x)$ is the frequency of the character $x$, and if $\text{length}(x)$ is the length of the code word corresponding to $x$, then the average number of bits per character will be: $\sum_{x \in C} f(x) \cdot \text{length}(x)$
- Also – $\sum_{x \in C} f(x) = 1$
- Just think of the values of the function $f$ as weights.
- Our problem is – given the set $C$ and the frequency function $f$, find a prefix encoding that minimizes this value.
Decoding

- Retrieval of original text.
- The codes can be represented by binary trees (left: fixed code. Right: variable code).
The depth of a leaf in the tree is just the length of the code word for that character.

Let $d_T(x)$ be the depth of a leaf node corresponding to the character $x$ in the tree $T$.

The average cost $AC$ per character in the encoding scheme defined by the tree $T$ is

$$AC(T) = \sum_{x \in C} f(x)d_T(x)$$
Exhaustive search:

- Enumerate all possible prefix trees and find the one with the smallest average cost per character.
- Without performing an exact analysis, the cost of this algorithm would be exponential in the number of characters, and therefore completely useless.
Finding the Optimal Encoding

**Lemma**

If $T$ is the tree corresponding to an optimal prefix encoding, and if $T_L$ and $T_R$ are its left and right subtrees, respectively, then $T_L$ and $T_R$ are also trees corresponding to optimal prefix encodings.

**Proof.**

- Let us say that $C_L$ is the set of characters that are leaf nodes in $T_L$ and similarly for $C_R$ and $T_R$.
- If $x \in C_L$, then certainly $d_{T_L}(x) = d_T(x) - 1$, and the same is true for $C_R$ and $T_R$. 
Therefore we can see from our basic cost formula that

\[ AC(T) = \sum_{x \in C} f(x)d_T(x) \]

\[ = \sum_{x \in C_L} f(x)(d_{T_L}(x) + 1) + \sum_{x \in C_R} f(x)(d_{T_R}(x) + 1) \]

\[ = \sum_{x \in C_L} f(x)d_{T_L}(x) + \sum_{x \in C_R} f(x)d_{T_R}(x) + \sum_{x \in C} f(x) \]

If \( T_R \) were not an optimal encoding tree, then we could replace it by a more efficient one (with the same leaves and the same frequencies), and this would show in turn that \( T \) could not have been optimal, a contradiction.
Corollary

If $T$ is the tree corresponding to an optimal prefix encoding, then every subtree of $T$ also corresponds to an optimal prefix encoding.

Proof.

This follows immediately by induction.

- This lemma expresses the fact that the problem of finding an optimal prefix code has the optimal substructure property.
- This means that we could write a recursive algorithm for it.
Finding the Optimal Encoding – Recursive Algorithm

- Start with a worklist consisting of $n$ trees, each tree consisting of exactly 1 character.
- From these trees construct other trees bottom-up and add them to the worklist.
- As each new tree is constructed, check the worklist to see if a tree with the same leaves is in it.
- Keep the tree with the smallest cost in the worklist and remove any others with the same set of leaves.
- At the end of this process there will be one tree in the worklist that contains all the characters in $C$ as leaves, and that tree represents an optimal encoding.
- This algorithm will definitely give the correct answer, but is still inefficient, although it is better than exhaustive search.
The optimal substructure property should remind us of dynamic programming.

If there were also an overlapping subproblems property of this problem, we could try such a solution.

Actually we have something even better: We don’t actually have to form all possible trees on the way up and check them all.

We actually can tell at each step exactly which tree to form.
Lemma

Let \( x \) and \( y \) be two characters in \( C \) having the lowest frequencies. Then there exists an optimal prefix code for \( C \) in which the codewords for \( x \) and \( y \) have the same length and differ only in the last bit.

Proof.

- Suppose that the tree \( T \) represents an optimal prefix code for our problem.
- If \( x \) and \( y \) are sibling nodes of greatest depth, then we are done.
- Otherwise, suppose that \( p \) and \( q \) are sibling nodes of greatest depth.
- We will exchange \( x \) and \( p \), and we will also exchange \( y \) and \( q \).
Proof (cont.)

- We know that

\[
\begin{align*}
  d_T(x) &\leq d_T(p) \\
  d_T(y) &\leq d_T(q) \\
  f(x) &\leq f(p) \\
  f(y) &\leq f(q)
\end{align*}
\]

- Suppose the tree $T$, after these two switches, is turned into the tree $T'$. Then we have:

\[
\begin{align*}
  d_{T'}(x) &= d_T(p) \\
  d_{T'}(p) &= d_T(x) \\
  d_{T'}(y) &= d_T(q) \\
  d_{T'}(q) &= d_T(y)
\end{align*}
\]
Proof (cont.)

\[ AC(T') - AC(T) = \sum_{z \in C} f(z)(d_{T'}(z) - d_T(z)) \]
\[ = f(p)(d_{T'}(p) - d_T(p)) + f(x)(d_{T'}(x) - d_T(x)) \]
\[ + f(q)(d_{T'}(q) - d_T(q)) + f(y)(d_{T'}(y) - d_T(y)) \]
\[ = f(p)(d_T(x) - d_T(p)) + f(x)(d_T(p) - d_T(x)) \]
\[ + f(q)(d_T(y) - d_T(q)) + f(y)(d_T(q) - d_T(y)) \]
\[ = (f(p) - f(x))(d_T(x) - d_T(p)) \]
\[ + (f(q) - f(y))(d_T(y) - d_T(q)) \]
\[ \leq 0 \]

so \( AC(T') \leq AC(T) \), which shows that \( T \) was not an optimal tree to begin with, and this is a contradiction.
Finding the Optimal Encoding – Huffman’s Algorithm

- We can start out with our initial worklist, and we can take two nodes of smallest frequency and build a tree from them (in which they are the two leaves).
- Then we delete those two nodes from the worklist, because we know that they will definitely be part of the little tree we have just constructed – we will never have to look at them again.
- By exactly the same argument, we can take the two elements of the worklist that are now of smallest cost, and build a little tree from them, and then throw them away.
- When we are done, we have the tree we are looking for.
- The algorithm: We keep a minimum-priority queue $Q$ of subtrees. $Q$ initially consists of the $n$ characters. The priority of any element in $Q$ will be the cost of that subtree.
**Algorithm 1** Huffman(C)

1: \( n \leftarrow |C| \)
2: \( Q \leftarrow C \)
3: **for** \( i \leftarrow 1 \ldots n - 1 \) **do**
4: allocate a new node \( z \)
5: \( \text{left}[z] \leftarrow \text{ExtractMin}(Q) \)
6: \( \text{right}[z] \leftarrow \text{ExtractMin}(Q) \)
7: \( f[z] \leftarrow f[x] + f[y] \)
8: \( \text{Insert}(Q.z) \)
9: **end for**
10: **return** \( \text{ExtractMin}(Q) \) \hspace{1cm} //Return the root of the tree.
This algorithm works even better than a dynamic programming algorithm: we don’t have to memoize intermediate results for later use.

We know exactly at each step what we need to do.

This is called a “greedy” algorithm because we chose the locally best solution at each step.

In effect, we act as if we were “greedy”.

What is is the best at each step is guaranteed (in this case) to turn to out to be the best overall.
Another Example – Activity Selection

**Input:** Set $S$ of $n$ activities – $\{a_1, a_2, \ldots, a_n\}$.

- $s_i =$ start time of activity $i$.
- $f_i =$ finish time of activity $i$.

**Output:** Subset $A$ of maximum number of compatible activities.

Two activities are compatible, if their intervals do not overlap.

Example (activities in each line are compatible):
Optimal Substructure

• Assume activities are sorted by finishing times – 
  \( f_1 \leq f_2 \leq \cdots \leq f_n \).
• Suppose an optimal solution includes activity \( a_k \).
• This generates two subproblems:
  • Selecting from \( a_1, \ldots, a_{k-1} \), activities compatible with one another, and that finish before \( a_k \) starts (compatible with \( a_k \)).
  • Selecting from \( a_{k+1}, \ldots, a_n \), activities compatible with one another, and that start after \( a_k \) finishes.
• The solutions to the two subproblems must be optimal.
• Prove using the cut-and-paste approach.
Let $S_{ij} =$ subset of activities in S that start after $a_i$ finishes and finish before $a_j$ starts.

Subproblems: Selecting maximum number of mutually compatible activities from $S_{ij}$.

Let $c[i,j] =$ size of maximum-size subset of mutually compatible activities in $S_{ij}$.

The recursive solution is:

$$c[i, j] = \begin{cases} 
0 & \text{if } S_{ij} = \emptyset \\
\max_{i < k < j} \{c[i, k] + c[k, j] + 1\} & \text{otherwise}
\end{cases}$$
The problem also exhibits the greedy-choice property.

There is an optimal solution to the subproblem $S_{ij}$, that includes the activity with the smallest finish time in set $S_{ij}$.

It can be proved easily (how?).

Hence, there is an optimal solution to $S$ that includes $a_1$.

Therefore, make this greedy choice without solving subproblems first and evaluating them.

Solve the subproblem that ensues as a result of making this greedy choice.

Combine the greedy choice and the solution to the subproblem.
Algorithm 2 Recursive-Activity-Selector (s, f, i, j)

1: \( m \leftarrow i + 1 \)
2: \( \textbf{while} \ m < j \text{ and } s_m < f_i \ \textbf{do} \)
3: \( m \leftarrow m + 1 \)
4: \( \textbf{end while} \)
5: \( \textbf{if} \ m < j \ \textbf{then} \)
6: \( \textbf{return} \ a_m \cup \text{Recursive} - \text{Activity} - \text{Selector}(s, f, m, j) \)
7: \( \textbf{else} \)
8: \( \textbf{return} \ \emptyset \)
9: \( \textbf{end if} \)

- Top level call: \( \text{Recursive} - \text{Activity} - \text{Selector}(s, f, 0, n + 1) \)
- Complexity??
- See text for iterative version
Typical Steps

- Cast the optimization problem as one in which we make a choice and are left with one subproblem to solve.
- Prove that there is always an optimal solution that makes the greedy choice, so that the greedy choice is always safe.
- Show that greedy choice and optimal solution to subproblem $\Rightarrow$ optimal solution to the problem.
- Make the greedy choice and solve top-down.
- May have to preprocess input to put it into greedy order.
- Example: Sorting activities by finish time.