1 Stirling’s formula

How fast does $n!$ grow? It’s immediately evident that for $n \geq 4$,

$$2^n \leq n! \leq n^n$$

and from this in turn we see that

$$n \leq \log_2 n! \leq n \log_2 n$$

This is nice, but we really want a tighter approximation.

Stirling’s formula is a very tight approximation indeed. In one of its simplest forms, it states that

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

In fact, one can even extend this:

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \Theta\left(\frac{1}{n^3}\right)\right)$$

and one can even go farther—there is a formula for the coefficients in the series on the right. For our purposes, however, we don’t need to go very far at all.

2 A simple application

In fact, from (1), we can see that

$$\log n! = \Theta(n \log n)$$

2.1 Exercise

1. Use the fact that

$$\log x = \int_1^x \frac{dt}{t}$$

to show that $0 \leq \log(1 + h) \leq h$ for all $h \geq 0$. (Hint: for $t \geq 1$, bound the function $1/t$ below and above by really simple bounds.)
2. Use the result you just proved, together with Stirling’s formula, to show that

\[ \log n! = \Theta(n \log n) \]

(3) is the approximation we need. Of course, this derivation depends on the proof of Stirling’s formula, which we are not going to prove here. However, if we don’t care about the full strength of Stirling’s formula but only want (3), we can derive it quite directly:

2.2 Exercise Show that

\[ \int_1^n \log t \, dt \leq \log n! \leq \int_1^{n+1} \log t \, dt \]

(Use approximations from above and below, just as we did when getting an approximation for the harmonic numbers \( H_n \). This time, use the function \( f(t) = \log t \).)

Then use your knowledge of these integrals to show that

\[ \log n! = \Theta(n \log n) \]