Centralizers of Nilpotents and The Bala-Carter Classification

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Let $\mathfrak{g}_c$ be a complex semisimple Lie algebra with adjoint group $G_c$ and $e$ a nilpotent element in $\mathfrak{g}_c$.

Let $G_c^e$ centralizer of $e$ in $G_c$. $G_c^e$ is not connected in general. Often it is required to compute the finite group

$$A(e) = \frac{G_c^e}{(G_c^e)_o}$$

where $(G_c^e)_o$ is the identity component.

E. Sommers (1997) gave a unified description of the conjugacy classes of $A(e)$. 
A pseudo-Levi subalgebra $l$ of $\mathfrak{g}_c$ is defined to be the centralizer in $\mathfrak{g}_c$ of a semisimple element $z$ of $G_c$. (Notation $l = \mathfrak{g}_c^z$)

Pseudo-Levi subgroups $L$ of $G_c$ are of the form $(G_c^z)$. the identity component of the centralizer of $z$ in $G_c$.

Let $Z$ be the center of $L$ in $G_c$ then the group $\frac{Z}{Z_o}$ is cyclic. (Sommers)

A nilpotent element $e$ of $\mathfrak{g}_c$ is called distinguished if the conditions $x \in \mathfrak{g}_c$ semisimple and $[x, e] = 0$ imply that $x$ is in the center of $\mathfrak{g}_c$.

Let $L \subset G_c$ be a pseudo-Levi subgroup with center $Z$ and Lie algebra $l$ a
pseudo-Levi subalgebra of $\mathfrak{g}_c$. Given a conjugacy class $\bar{c}$ of $A(e)$, $l$ has the key property for $(e, \bar{c})$ if $e \in l$ and there exists $z \in Z$ such that

1. $zZ_\circ$ generates the cyclic group $\frac{Z}{Z_\circ}$

2. $z(G_c^e)_\circ = \bar{c}$

Moreover if $l$ is a minimal pseudo-Levi subalgebra with the key property for $(e, \bar{c})$ then $e$ is distinguished in $l$.

Sommers shows how to build minimal pseudo-Levi subalgebras with the key property for a given pair $(e, \bar{c})$. 


Theorem (Sommers). There is a bijection $\Phi$ between $G_c$ conjugacy classes of pairs $(l, e)$, where $l$ is a pseudo-Levi subalgebra and $e$ is a distinguished nilpotent in $l$, and $G_c$ conjugacy classes of pair $(e, \bar{c})$, where $e$ is a nilpotent element in $\mathfrak{g}_c$ and $\bar{c}$ is a conjugacy class of $A(e)$.

Example. Let $\mathfrak{g}_c$ be $\mathfrak{so}_5(\mathbb{C})$, the complex Lie algebra of $5 \times 5$ orthogonal matrices. It is known that its nilpotent classes are parametrized by certain partitions of 5. Consider the class $[3,1,1]$. A computation shows that the only two pseudo-Levi subalgebras (up to conjugacy) containing a nilpotent $e$ in that class are of the form $\mathfrak{sl}_2(\mathbb{C})$ and $2\mathfrak{sl}_2(\mathbb{C})$. 
Sommer’s Correspondence

\[(e, 1) \Leftrightarrow (\mathfrak{sl}_2(\mathbb{C}), e)\]

\[(e, -1) \Leftrightarrow (2\mathfrak{sl}_2(\mathbb{C}), e)\]

In this case \(A(e) = \mathbb{Z}_2\).

Sommer’s theorem is an extension of the Bala-Carter classification which states:

The nilpotent orbits \(G_c.e\) of \(\mathfrak{g}_c\) are in bijection with \(G_c\) conjugacy classes of pairs \((l, p_l)\) where \(l\) is a Levi subalgebra of \(\mathfrak{g}_c\) in which \(e\) is distinguished and \(p_l\) a distinguished parabolic subalgebra of \([l, l]\).

In fact \((e, 1)\) always goes to the B-C Levi \(l\).
Our goal is to extend Sommer’s result to real reductive Lie groups.

Let $g$ be a real reductive Lie algebra with adjoint group $G$ and Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$ relative to a Cartan involution $\Theta$.

We complexify $g$ to obtain $g_c = \mathfrak{k}_c \oplus \mathfrak{p}_c$.

Denote by $\sigma$ the conjugation of $g_c$ with regard to $g$.

Let $K_c$ be the complexification of $K$ the connected subgroup of $G$ with Lie algebra $\mathfrak{k}$. $K_c$ preserves $\mathfrak{p}_c$ under the adjoint action.
Sekiguchi proves that there is a one to one correspondence between nilpotent $G$-orbits in $\mathfrak{g}$ and nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$.

Therefore it is reasonable to solve the following problem.

Let $e$ be a nilpotent element in $\mathfrak{p}_{\mathbb{C}}$ and $A_k(e) = \frac{K_{\mathbb{C}}^e}{(K_{\mathbb{C}}^e)^o}$. Give a unified description of the elements of $A_k(e)$.

A pseudo-Levi subalgebra $l$ of $\mathfrak{g}_{\mathbb{C}}$ is said to be \textbf{elliptic} if it is the centralizer in $\mathfrak{g}_{\mathbb{C}}$ of an elliptic element of $K_{\mathbb{C}}$. 
The connected subgroup $L$ of $G_\mathbb{C}$ with Lie algebra $l$ is an elliptic pseudo Levi subgroup of $G_\mathbb{C}$. Let $Z$ be the center of $L$ then:

**Non trivial Fact:**

\[
\frac{Z\cap K_\mathbb{C}}{(Z\cap K_\mathbb{C})^0} \text{ is cyclic}
\]

A nilpotent element $e$ of $l \cap p_\mathbb{C}$ is called **noticed** if the conditions $z \in l \cap \mathfrak{k}_\mathbb{C}$ semisimple and $[z, e] = 0$ imply that $z$ is in the center of $l$.

We can choose $e$ such that it lies in a Kostant-Sekiguchi $\mathfrak{sl}_2$—triple $\{x, e, f\}$ that is $x \in \mathfrak{k}_\mathbb{C}$, $e$ and $f$ in $p_\mathbb{C}$, $\sigma(e) = f$. 
A $\theta$-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{l}$ is said to be noticed for $e$ if there is a K-S triple in $\mathfrak{l}$ for which $\mathfrak{q}$ is the sum of the non-negative eigenspaces of $ad(x)$ acting on $\mathfrak{l}$.

Given a conjugacy class $\bar{c}$ of $A_k(e)$, $\mathfrak{l}$ has the **key property** for $(e, \bar{c})$ if $e \in \mathfrak{l}$ and there exits $z \in Z \cap K_{\bar{c}}$ such that

1. $z(Z \cap K_{\bar{c}})_o$ generates $\frac{Z \cap K_{\bar{c}}}{(Z \cap K_{\bar{c}})_o}$

2. $z(K^e_{\bar{c}})_o = \bar{c}$

Moreover if $\mathfrak{l}$ is a minimal elliptic pseudo-Levi subalgebra with the key property for $(e, \bar{c})$ then $e$ is noticed in $\mathfrak{l}$. 
We know how to build minimal elliptic pseudo-Levi subalgebras with the key property for a given pair \((e, \bar{c})\).

**Theorem (1999).** There is a one to one correspondence between \(K_c\)-conjugacy classes of pairs \((e, \bar{z})\), where \(e\) is a nilpotent in \(\mathfrak{p}_c\), \(\bar{z} \in A_k(e)\) and \(K_c\)-conjugacy classes of triples \((l, q_l, w)\) where \(l\) is an elliptic pseudo-Levi subalgebra in which \(e\) is noticed, \(q_l\) is a noticed parabolic of \(l\) for \(e\), and \(w\) is a certain prehomogeneous space.
Example 1. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$. The two non-zero nilpotent orbits of $\mathfrak{g}$ are parametrized by the matrices

$$e_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A computation gives:

$$A_k(e_1) = id \text{ and } A_k(e_2) = \mathbb{Z}_2$$

We obtain the following correspondence:

$$(e_1, id) \Leftrightarrow (\mathfrak{sl}_3(\mathbb{C}), b_1, w_1)$$

$$(e_2, id) \Leftrightarrow (\mathfrak{sl}_3(\mathbb{C}), b_2, w_2)$$

$$(e_2, -id) \Leftrightarrow (\mathfrak{sl}_2(\mathbb{C}) \oplus V, b_3, w_3)$$
Example 2. Let $g = \mathfrak{so}(3, 2)$. The complex nilpotent orbit $[3, 1, 1]$ splits into three classes $e_1, e_2, e_3$ in $g$.

\[
\begin{array}{cccc}
- & + & - & + \\
+ & & + & \\
+ & & - & \\
e_1, e_2 & & e_3 & \\
\end{array}
\]

\[
(e_1, id) \Leftrightarrow (\mathfrak{sl}_2(\mathbb{C}), b_1, w_1)
\]

\[
(e_1, -id) \Leftrightarrow (2\mathfrak{sl}_2(\mathbb{C}), b_2, w_2)
\]

\[
(e_2, id) \Leftrightarrow (\mathfrak{sl}_2(\mathbb{C}), b_3, w_3)
\]

\[
(e_2, -id) \Leftrightarrow (2\mathfrak{sl}_2(\mathbb{C}), b_4, w_4)
\]

\[
(e_3, id) \Leftrightarrow (\mathfrak{so}_5(\mathbb{C}), q, w_5)
\]

\[
(e_3, -id) \Leftrightarrow (2\mathfrak{sl}_2(\mathbb{C}), b_5, w_6)
\]
This theorem is an extension of the our classification which states:

The nilpotent orbits $K_C.e$ of $g_C$ are in bijection with $K_C$-conjugacy classes of triples $(l, q_l, w_l)$ where $l$ is a $(\Theta, \sigma)$ Levi subalgebra of $g_C$ in which $e$ is noticed, $q_l$ a noticed parabolic subalgebra of $[l, l]$ and $w_l$ is an $L \cap K_C$-module in $l$.

In fact $(e, 1)$ always goes to this $(\Theta, \sigma)$-stable Levi $l$.
REFERENCES:


