Homework 2

Posted: February 21, 2024
Due: March 6, 2024

1. Let $S_{\times}^{n \times n}$ be the set of $n \times n$ symmetric matrices in $\mathbb{R}^{n \times n}$. Prove that $S_{\times}^{n \times n}$ is a subspace of $\mathbb{R}^{n \times n}$ and $\dim(S_{\times}^{n \times n}) = \frac{n(n+1)}{2}$.

2. Let $A, B \in \mathbb{C}^{\times n}$ be two matrices. Prove that if $AB = BA$, then we have the following equality known as Newton’s binomial:

$$(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k.$$ 

Give an example of matrices $A, B \in \mathbb{C}^{2 \times 2}$ such that $AB \neq BA$ for which the above formula does not hold.

3. Let $M \in \mathbb{R}^{\times n}$ be a partitioned matrix,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathbb{R}^{m \times m}$ and $m < n$. Prove that, if all inverse matrices mentioned next exist and $Q = (A - BD^{-1}C)^{-1}$, then

$$M^{-1} = \begin{pmatrix} Q & -QBD^{-1} \\ -D^{-1}CQ & D^{-1} + D^{-1}CQBD^{-1} \end{pmatrix}.$$ 

4. Prove or disprove:

(a) If $0 \in W$, where $W = \{w_1, \ldots, w_n\}$, then $W$ is linearly independent.

(b) If $W = \{w_1, \ldots, w_n\}$ is linearly independent and $w$ is not a linear combination of the vectors of $W$, then $W \cup \{w\}$ is linearly independent.
(c) If \( W = \{w_1, \ldots, w_n\} \) is linearly dependent, then any of \( w_i \) is a linear combination of the others.

(d) If \( y \) is not a linear combination of \( \{w_1, \ldots, w_n\} \), then \( \{y, w_1, \ldots, w_n\} \) is linearly independent.

(e) If any \( n - 1 \) vectors of the set \( W = \{w_1, \ldots, w_n\} \) are linearly dependent, then \( W \) is linearly independent.

5. An involutive matrix is a matrix \( A \in \mathbb{C}^{n \times n} \) such that \( A^2 = I_n \). An idempotent matrix is a matrix \( B \in \mathbb{C}^{n \times n} \) such that \( B^2 = B \).

Prove that if \( B \in \mathbb{C}^{n \times n} \) is an idempotent matrix, then \( A = 2B - I_n \) is an involutive matrix.