1. Let $S_{n \times n}$ be the set of $n \times n$ symmetric matrices in $\mathbb{R}^{n \times n}$. Prove that $S_{n \times n}$ is a subspace of $\mathbb{R}^{n \times n}$ and $\dim(S_{n \times n}) = \frac{n(n+1)}{2}$.

2. Let $A, B \in \mathbb{C}^{n \times n}$ be two matrices. Prove that if $AB = BA$, then we have the following equality known as Newton’s binomial:

$$ (A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k}B^k. $$

Give an example of matrices $A, B \in \mathbb{C}^{2 \times 2}$ such that $AB \neq BA$ for which the above formula does not hold.

3. Let $M \in \mathbb{R}^{n \times n}$ be a partitioned matrix,

$$ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, $$

where $A \in \mathbb{R}^{m \times m}$ and $m < n$. Prove that, if all inverse matrices mentioned next exist and $Q = (A - BD^{-1}C)^{-1}$, then

$$ M^{-1} = \begin{pmatrix} Q & -QBD^{-1} \\ -D^{-1}CQ & D^{-1} + D^{-1}CQBD^{-1} \end{pmatrix}. $$

4. Prove or disprove:
   (a) If $\mathbf{0} \in W$, where $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$, then $W$ is linearly independent.
   (b) If $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is linearly independent and $\mathbf{w}$ is not a linear combination of the vectors of $W$, then $W \cup \{\mathbf{w}\}$ is linearly independent.
(c) If $W = \{w_1, \ldots, w_n\}$ is linearly dependent, then any of $w_i$ is a linear combination of the others.

(d) If $y$ is not a linear combination of $\{w_1, \ldots, w_n\}$, then $\{y, w_1, \ldots, w_n\}$ is linearly independent.

(e) If any $n - 1$ vectors of the set $W = \{w_1, \ldots, w_n\}$ are linearly dependent, then $W$ is linearly independent.

5. An involutive matrix is a matrix $A \in \mathbb{C}^{n \times n}$ such that $A^2 = I_n$.
   Prove that if $B \in \mathbb{C}^{n \times n}$ is an idempotent matrix, then $A = 2B - I_n$ is an involutive matrix.