CS724: Topics in Algorithms
Problem Set 1

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Problem 1:

Prove that if $m, n \in \mathbb{N}$ we have $\equiv_n \subseteq \equiv_m$ if and only if $m$ evenly divides $n$. 
Solution 1:

If $m$ is a divisor of $n$ we have $n = km$. Thus, if $p - q$ is divisible by $n$, it is also divisible by $m$, hence $p \equiv_m q$.

Conversely, suppose that $\equiv_n \subseteq \equiv_m$. Since $(0, n) \in \equiv_n$, it follows that $(0, n) \in \equiv_m$, which means that $n = n - 0 = km$. Thus, $m$ evenly divides $n$. 
Let \( p_1, p_2, p_3, \ldots \) be the sequence of prime numbers \( 2, 3, 5, \ldots \). Define the function \( f : \text{Seq}(\mathbb{N}) \rightarrow \mathbb{N} \) by \( f(n_1, \ldots, n_k) = p_1^{n_1} \cdots p_k^{n_k} \). Prove that \( f(n_1, \ldots, n_k) = f(m_1, \ldots, m_k) \) implies \( (n_1, \ldots, n_k) = (m_1, \ldots, m_k) \).
Solution 2: Suppose that \( f(n_1, \ldots, n_k) = f(m_1, \ldots, m_k) \), that is,

\[
p_1^{n_1} \cdots p_k^{n_k} = p_1^{m_1} \cdots p_k^{m_k}.
\]

Note that this implies \( p_i^{n_i} | p_i^{m_i} \) and \( p_i^{m_i} | p_i^{n_i} \), or \( n_i \leq m_i \) and \( m_i \leq n_i \). Thus, \( n_i = m_i \) for \( 1 \leq i \leq k \).
Problem 3:

Let $\mathcal{C}$ be a collection of subsets of a set $S$ such that $\bigcup \mathcal{C} = S$. Define the relation $\rho_\mathcal{C}$ on $S$ by

$$
\rho_\mathcal{C} = \{(x, y) \in S \times S \mid x \in C \text{ if and only if } y \in C \text{ for every } C \in \mathcal{C}\}.
$$

Prove that, for every collection $\mathcal{C}$, the relation $\rho_\mathcal{C}$ is an equivalence. What changes if the condition $\bigcup \mathcal{C} = S$ is not satisfied?
Solution 3:

ρ_C is clearly reflexive because for each x ∈ S there exists C ∈ C such that x ∈ C. Suppose that (x, y) ∈ ρ_C. Then, we have x ∈ C if and only if y ∈ C hence (y, x) ∈ ρ. Thus, ρ_C is symmetric. If (x, y), (y, z) ∈ ρ_C x ∈ C if and only if y ∈ C and y ∈ C if and only if z ∈ C for any C ∈ C. Thus, (x, z) ∈ ρ_C, hence ρ is a transitive relation.
Let $S$ and $T$ be two finite sets such that $|S| = m$ and $|T| = n$.

- Prove that the set of functions $S \rightarrow T$ contains $n^m$ elements.
- Prove that the set of partial functions $S \leadsto T$ contains $(n + 1)^m$ elements.
A function $f : S \rightarrow T$ can be represented by an array having $m$ locations corresponding to the elements $x_1, \ldots, x_n$ of the set $S$. If we write in the box that corresponds to $x_i$ the value of $f(x_i)$ we have $n$ choices for each of the boxes. Such an array corresponds to a function $f : S \rightarrow T$, so we have $n^m$ functions.

$$
\begin{array}{cccc}
  f(x_1) & f(x_2) & \cdots & f(x_m) \\
 x_1 & x_2 & \cdots & x_m \\
\end{array}
$$

A partial function can be represented by a similar array. If $f(x_i)$ is defined, we write the value $f(x_i)$ in the box; if not, we write ↑. Thus, there are $n + 1$ choices for each box, and the total number of partial functions is $(n + 1)^m$. 
Problem 5

Let $\mathcal{C}$ be a nonempty collection of nonempty subsets of a set $S$. Prove that $\mathcal{C}$ is a partition of $S$ if and only if every element $a \in S$ belongs to exactly one member of the collection $\mathcal{C}$. 
Let $A, B$ be two sets in $\mathcal{C}$. Note that if $x \in A \cap B$, then $x \in A$ and $x \in B$, and this violates the description of $\mathcal{C}$. Thus, any two sets in $\mathcal{C}$ are disjoint. Since every $a \in S$ belongs to a set of $\mathcal{C}$, the union of these sets equals $S$, hence $\mathcal{C}$ is a partition.
Things to remember when you do homework:

- write neatly, using latex;
- use clear and correct English;
- do not use the expression “it is easy to see”; fully justify your statements.