CS724: Topics in Algorithms
Problem Set 1

Prof. Dan A. Simovici
Problem 1:

The set of polynomials over a field $\mathbb{F}$, $\mathbb{F}[x]$, consists of all functions $f : \mathbb{F} \rightarrow \mathbb{F}$ of the form $f(x) = c_0 + c_2 x + \cdots + c_n x^n$, where $c = 0$, $c_1, \ldots, c_n$ are fixed scalars from $\mathbb{F}$. Prove that the set $\{x, x^2\}$ is linearly independent in $\mathbb{R}[x]$. 
Solution 1:

Suppose that \( ax + bx^2 = 0 \) for \( x \in \mathbb{R} \), where \( 0(x) = 0 \) for \( x \in \mathbb{R} \). This holds for every \( a \) and \( b \), so we can write

\[
\begin{align*}
a + b &= 0 \text{ (by taking } x = 1) \\
-a + b &= 0 \text{ (by taking } x = -1),
\end{align*}
\]

which implies \( a = b = 0 \).
Problem 2:

Prove that the set of complex numbers $\mathbb{C}$ can be regarded as a linear space over the field $\mathbb{R}$ of real numbers.
Solution 2: Let $u = a + ib$ and $v = c + id$ be two complex numbers. Their sum is $u + v = (a + c) + i(b + d)$; for $\alpha \in \mathbb{R}$, the product $\alpha u$ is $\alpha u = \alpha a + i\alpha b$. The addition and multiplication of complex numbers satisfy the definition of a complex space over $\mathbb{R}$. 
Let $W_1, W_2$ be subspaces of a real linear space $V$ such that the set union $W_1 \cup W_2$ is also a subspace. Prove that one of the subspaces $W_i$ is included in the other.
Solution 3:

Suppose that $W_1 \cup W_2$ is a subspace but neither subspace is contained in the other. Then, there exist $x \in W_1 - W_2$ and $y \in W_2 - W_1$. We claim that $x + y$ cannot be in either subspace, hence it cannot be in their union $W_1 \cup W_2$, which is a contradiction.

If $x + y \in W_1$, then $(x + y) - x \in W_1$, but this is $y$ and we have a contradiction. Similarly, $x + y$ cannot belong to $W_2$. 
Let $V$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$; let $V_{\text{even}}$ be the subset of all even functions, $f(x) = f(-x)$; let $V_{\text{odd}}$ be the subset of all odd functions, $f(-x) = -f(x)$.

Prove that:

- $V_{\text{even}}$ and $V_{\text{odd}}$ are subspaces of $V$;
- $V_{\text{even}} + V_{\text{odd}} = V$;
- $V_{\text{even}} \cap V_{\text{odd}} = \{0\}$.
Suppose that \( f, g \in V_{even} \), that is \( f(x) = f(-x) \) and \( g(x) = g(-x) \). Then

\[
(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x)
\]

so \( f + g \) is even. Also, \( (af)(x) = af(x) = af(-x) \), so \( V_{even} \) is a subspace. A similar argument works for \( V_{odd} \).
Solution 4 cont’d

If \( h \in V \) we can write \( h \) as

\[
h(x) = \frac{h(x) + h(-x)}{2} + \frac{h(x) - h(-x)}{2}
\]

The first function is even, and the second is odd, so \( V_{even} + V_{odd} = V \).

If \( f \in V_{even} \cap V_{odd} \) we have both \( f(x) = f(-x) \) and \( f(-x) = -f(x) \).

Therefore, \( f(x) = 0 \), and \( f \) is 0.
Let $W_1$ and $W_2$ be subspaces of a vector space $V$ such that

$$W_1 + W_2 = \{ u + v \mid u \in W_1, v \in W_2 \} = V,$$

and $W_1 \cap W_2 = \{ 0 \}$. Prove that each vector $v$ in $V$ can be uniquely written as a sum $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. 
Solution 5

By the definition of \( W_1 + W_2 \) it is clear that each vector \( v \) in \( V \) can be written as a sum \( v = w_1 + w_2 \), where \( w_1 \in W_1 \) and \( w_2 \in W_2 \). What needs to be shown is the uniqueness part.

Suppose that \( v \) can be written as:

\[
v = w_1 + w_2 = \tilde{w}_1 + \tilde{w}_2,
\]

where \( w_1, \tilde{w}_1 \in W_1 \) and \( w_2, \tilde{w}_2 \in W_2 \).

Since \( w_1 - \tilde{w}_1 = \tilde{w}_2 - w_2 \), \( w_1 - \tilde{w}_1 \in W_1 \), \( \tilde{w}_2 - w_2 \in W_2 \) it follows that these vector differences belong to \( W_1 \cap W_2 = \{0\} \), which means that \( w_1 - \tilde{w}_1 = 0 \) and \( \tilde{w}_2 - w_2 = 0 \). Thus, \( w_1 = \tilde{w}_1 \) and \( \tilde{w}_2 = w_2 \), which proves the uniqueness.
Things to remember when you do homework:

- name your file as “hw1-John.Doe.pdf”; this would allow me to recognize your file in the mail;
- write neatly, using latex;
- use clear and correct English;
- do not use the expression “it is easy to see”; fully justify your statements.