CS724: Topics in Algorithms
Problem Set 1

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Problem 1:

The set of polynomials over a field $\mathbb{F}$, $\mathbb{F}[x]$, consists of all functions $f : \mathbb{F} \longrightarrow \mathbb{F}$ of the form $f(x) = c_0 + c_2x + \cdots + c_nx^n$, where $c \neq 0$, $c_1, \ldots, c_n$ are fixed scalars from $\mathbb{F}$. Prove that the set $\{x, x^2\}$ is linearly independent in $\mathbb{R}[x]$. 
Solution 1:

Suppose that $ax + bx^2 = 0$ for $x \in \mathbb{R}$, where $0(x) = 0$ for $x \in \mathbb{R}$. This holds for every $a$ and $b$, so we can write

$$a + b = 0 \text{ (by taking } x = 1)$$
$$-a + b = 0 \text{ (by taking } x = -1),$$

which implies $a = b = 0$. 

Problem 2:

Prove that the set of complex numbers $\mathbb{C}$ can be regarded as a linear space over the field $\mathbb{R}$ of real numbers.
Solution 2: Let \( u = a + ib \) and \( v = c + id \) be two complex numbers. Their sum is \( u + v = (a + c) + i(b + d) \); for \( \alpha \in \mathbb{R} \), the product \( \alpha u \) is \( \alpha u = \alpha a + i\alpha b \). The addition and multiplication of complex numbers satisfy the definition of a complex space over \( \mathbb{R} \).
Problem 3:

Let $W_1, W_2$ be subspaces of a real linear space $V$ such that the set union $W_1 \cup W_2$ is also a subspace. Prove that one of the subspaces $W_i$ is included in the other.
Solution 3:

Suppose that $W_1 \cup W_2$ is a subspace but neither subspace is contained in the other. Then, there exist $x \in W_1 - W_2$ and $y \in W_2 - W_1$. We claim that $x + y$ cannot be in either subspace, hence it cannot be in their union $W_1 \cup W_2$, which is a contradiction.

If $x + y \in W_1$, then $(x + y) - x \in W_1$, but this is $y$ and we have a contradiction. Similarly, $x + y$ cannot belong to $W_2$. 
Problem 4

Let $V$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$; let $V_{even}$ be the subset of all even functions, $f(x) = f(-x)$; let $V_{odd}$ be the subset of all odd functions, $f(-x) = -f(x)$.

Prove that:

- $V_{even}$ and $V_{odd}$ are subspaces of $V$;
- $V_{even} + V_{odd} = V$;
- $V_{even} \cap V_{odd} = \{0\}$. 
Solution 4

Suppose that \( f, g \in V_{even} \), that is \( f(x) = f(-x) \) and \( g(x) = g(-x) \). Then

\[
(f + g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f + g)(-x)
\]

so \( f + g \) is even. Also, \((af)(x) = af(x) = af(-x)\), so \( V_{even} \) is a subspace. A similar argument works for \( V_{odd} \).
If $h \in V$ we can write $h$ as

$$h(x) = \frac{h(x) + h(-x)}{2} + \frac{h(x) - h(-x)}{2}$$

The first function is even, and the second is odd, so $V_{even} + V_{odd} = V$. If $f \in V_{even} \cap V_{odd}$ we have both $f(x) = f(-x)$ and $f(-x) = -f(x)$. Therefore, $f(x) = 0$, and $f$ is 0.
Let $W_1$ and $W_2$ be subspaces of a vector space $V$ such that

$$W_1 + W_2 = \{ u + v \mid u \in W_1, v \in W_2 \} = V,$$

and $W_1 \cap W_2 = \{0\}$. Prove that each vector $v$ in $V$ can be *uniquely* written as a sum $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. 


Solution 5

By the definition of $W_1 + W_2$ it is clear that each vector $v$ in $V$ can be written as a sum $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. What needs to be shown is the uniqueness part.

Suppose that $v$ can be written as:

$$v = w_1 + w_2 = \tilde{w}_1 + \tilde{w}_2,$$

where $w_1, \tilde{w}_1 \in W_1$ and $w_2, \tilde{w}_2 \in W_2$.

Since $w_1 - \tilde{w}_1 = \tilde{w}_2 - w_2$, $w_1 - \tilde{w}_1 \in W_1$, $\tilde{w}_2 - w_2 \in W_2$ it follows that these vector differences belong to $W_1 \cap W_2 = \{0\}$, which means that $w_1 - \tilde{w}_1 = 0$ and $\tilde{w}_2 - w_2 = 0$. Thus, $w_1 = \tilde{w}_1$ and $\tilde{w}_2 = w_2$, which proves the uniqueness.
Things to remember when you do homework:

- name your file as “hw1-John.Doe.pdf”; this would allow me to recognize your file in the mail;
- write neatly, using latex;
- use clear and correct English;
- do not use the expression “it is easy to see”; fully justify your statements.