1 Problems

2 Solutions
1. Let $\mathcal{P}$ be the program

IF $X \neq 0$ GOTO A

[A] $X \leftarrow X + 1$

IF $X \neq 0$ GOTO A

[A] $Y \leftarrow Y + 1$

What is $\Psi_{(1)}^{(1)}(x)$?

2. Show that for every partially computable function $f(x_1, \ldots, x_n)$ there is a number $m \geq 0$ such that $f$ is computed by infinitely many programs of length $m$.

3. Show by constructing a program that the predicate $x_1 \leq x_2$ is computable.

4. Let $P(x)$ be a computable predicate. Show that the function $E_P(r)$ defined by

$$E_P(r) = \begin{cases} 
1 & \text{if there are at least } r \text{ numbers } n \text{ such that } P(n) = 1 \\
\uparrow & \text{otherwise}
\end{cases}$$

is partially computable.
1. Let $\mathcal{P}$ be the program

\[
\text{IF } X \neq 0 \text{ GOTO } A \\
[A] \quad X \leftarrow X + 1 \\
\text{IF } X \neq 0 \text{ GOTO } A \\
[A] \quad Y \leftarrow Y + 1
\]

What is $\Psi_{\mathcal{P}}^{(1)}(x)$?

Solution: Note that regardless of the initial value of $X$, the statement $X \leftarrow X + 1$ is repeated indefinitely. Thus, $\Psi_{\mathcal{P}}^{(1)}(x_1) \uparrow$ for every $x \in \mathbb{N}$, hence $\Psi_{\mathcal{P}}^{(1)}(x_1) = \emptyset$. 
2. Show that for every partially computable function $f(x_1, \ldots, x_n)$ there is a number $m \geq 0$ such that $f$ is computed by infinitely many programs of length $m$.

Solution: If $\mathcal{P}$ is a program that computes $f$, let $Z_k$ be the first intermediate variable that does not occur in $\mathcal{P}$. Suppose that the length of $\mathcal{P}$ is $\ell$.

Consider the program $\mathcal{R}_h$ that has the length $m = \ell + 1$:

$$Z_h \leftarrow Z_h$$

$\mathcal{P}$

The family of programs $\{\mathcal{R}_h \mid h \geq k\}$ is infinite and all these programs compute $f$. 
3. Show by constructing a program that the predicate $x_1 \leq x_2$ is computable.

Solution:

$$
\begin{align*}
[L] & \quad X_1 \leftarrow X_1 - 1 \\
       & \quad X_2 \leftarrow X_2 - 1 \\
       & \quad \text{IF } X_1 \neq 0 \text{ GOTO } L_1 \\
       & \quad \text{IF } X_2 \neq 0 \text{ GOTO } L_2 \\
       & \quad \text{GOTO } L \\
[L_1] & \quad Y \leftarrow 1 \\
       & \quad \text{GOTO } E \\
[L_2] & \quad Y \leftarrow 0
\end{align*}
$$
4. Let $P(x)$ be a computable predicate. Show that the function $E_P(r)$ defined by

$$E_P(r) = \begin{cases} 1 & \text{if there are at least } r \text{ numbers } n \text{ such that } P(n) = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

is partially computable.

Solution:

\[
\begin{align*}
Z_1 & \leftarrow 0 \\
[L] & \text{ IF } Z_1 \geq r \text{ GOTO } L_2 \\
& \text{ IF } \sim P(Z_1) \text{ GOTO } L_1 \\
& \text{ Z}_1 \leftarrow Z_1 + 1 \\
[L_1] & \text{ GOTO } L \\
[L_2] & Y \leftarrow 1
\end{align*}
\]