1 Problems
Problem 1: For $A, B \subseteq \mathbb{N}$ define the set $A + B$ as

$$A + B = \{ x + y \mid x \in A \text{ and } y \in B \}.$$ 

Prove that if $A$ and $B$ are r.e., then $A + B$ is r.e.

Solution for Problem 1: Since both $A$ and $B$ are r.e., by the Projection theorem, there exist two primitive recursive predicate $R_A(x, t)$ and $R_B(x, t)$ such that:

$$A = \{ x \mid (\exists t)R_A(x, t) \},$$
$$B = \{ x \mid (\exists t)R_B(x, t) \}.$$
Therefore, we have

\[ z \in A + B \iff (\exists x)(\exists y)[z = x + y \& x \in A \& y \in B] \]
\[ \iff (\exists x)(\exists y)[z = x + y \& (\exists u)R_A(x, u) \& (\exists v)R_B(y, v)]. \]

Encode the four variables as \( t = [x, y, u, v] \), so \( Lt(t) = 4 \), 
(\( t \))_1 = x, (\( t \))_2 = y, (\( t \))_3 = u, and (\( t \))_4 = v.
Solution for Problem 1 cont’d

Thus,

\[ z \in A + B \iff (\exists t)[(Lt(t) = 4) \& z = (t)_1 + (t)_2 \]
\[ & \& R_A((t)_1, (t)_3) \& R_B((t)_2, (t)_4)] \].

This allows us to write

\[ A + B = \{ z \mid (\exists t)R_{A+B}(z, t) \} , \]

where

\[ R_{A+B}(z, t) = (\exists t)[(Lt(t) = 4) \& z = (t)_1 + (t)_2 \]
\[ & \& R_A((t)_1, (t)_3) \& R_B((t)_2, (t)_4)] , \]

which shows that \( A + B \) is r.e.
Problem 2: Let $\text{FINITE} = \{x \in \mathbb{N} \mid W_x \text{ is finite}\}$. Prove that $\overline{K} \leq_m \text{FINITE}$.

Solution for Problem 2: Consider the function

$$g(y, x) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that if $x \in K$, the function $g(y, x)$ is defined for every $y$ and hence not in $\text{FINITE}$.

If $x \in \overline{K}$, $g(y, x)$ is defined nowhere, and therefore, is in $\text{FINITE}$.

In either case, $g$ is partially computable because $K$ is recursive.
Solution for Problem 2 cont’d:
Let $p$ the number of a program that computes $g$. By the smn Theorem we have:

$$\Phi^{(2)}(y, x, p) = \Phi^{(1)}(y, S^1_1(x, p)) = \Phi^{(1)}(y, f(x)),$$

where $f(x) = S^1_1(x, p)$.

Then, we have:

$$x \in \overline{K} \Rightarrow (\forall y)[g(y, x) \uparrow]$$
$$\Rightarrow (\forall y)[\Phi^{(2)}(y, x, p) \uparrow]$$
$$\Rightarrow (\forall y)[\Phi^{(1)}(y, S^1_1(x, p)) \uparrow]$$
$$\Rightarrow (\forall y)[\Phi^{(1)}(y, f(x)) \uparrow]$$
$$\Rightarrow \mathcal{W}_{f(x)} = \emptyset$$
$$\Rightarrow f(x) \in \text{FINITE}.$$
Solution for Problem 2 cont’d:
On the other hand, we have:

\[ x \notin K \Rightarrow x \in K \]
\[ \Rightarrow (\forall y)[g(y, x) = y] \]
\[ \Rightarrow (\forall y)[\Phi^{(1)}(y, f(x)) = y] \]
\[ \Rightarrow f(x) \text{ is the number of a program for the identity function} \]
\[ \Rightarrow W_{f(x)} = \mathbb{N} \]
\[ \Rightarrow f(x) \notin \text{FINITE}. \]

Therefore, \( x \in \overline{K} \) if and only if \( f(x) \in \text{FINITE} \), hence \( \overline{K} \leq_m \text{FINITE} \).

Thus, there is no program that can test if an arbitrary program is defined on a finite number of input values.
Problem 3: Let $R_1$ be the set:

$$R_1 = \{ x \mid \text{range of } \Phi_x \text{ is finite} \}.$$

Determine whether the set $R_1$ is recursive, r.e. but not recursive, or not r.e.

Solution for Problem 3:

Let $\Gamma$ be the collection of functions

$$\Gamma = \{ \Phi_x \mid \text{range of } \Phi_x \text{ is finite} \},$$

hence $R_1$ is the index set of $\Gamma$. $R_1$ is not r.e. Indeed, let $f$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \uparrow & \text{otherwise}. \end{cases}$$

Since $\text{range}(f) = \{0\}$, $f \in \Gamma$. The identity function $g(x) = x$ is an extension of $g$ and it does not belong to $\Gamma$ because its range is $\mathbb{N}$ and this is not finite. By the other Rice’s theorem, $R_1$ is not r.e., and therefore is not recursive either.
Problem 4: Let $R_2$ be the set:

$$R_2 = \{ x \mid \text{range of } \Phi_x \text{ is } \mathbb{N} \}.$$ 

Determine whether the set $R_2$ is recursive, r.e. but not recursive, or not r.e.

Solution for Problem 4:
Let $x$ be a program such that $\text{range}(\Phi_x) = \mathbb{N}$. Any restriction of $\Phi_x$ to a finite subset can only produce a function with a finite range because a function can only map a finite set to another finite set. As a result, any finite restriction of $\Phi_x$ cannot have the program number in $R_2$, so $R_2$ is not r.e.
Problem 5: Let $a$ be a fixed number and let $R_a = \{x \mid a \not\in \text{range}(\Phi_x)\}$. Prove that $R_a$ is not r.e.

Solution for Problem 5:
Let $\Gamma$ be the set of programs whose ranges do not contain the fixed value $a$. Then, $R_a$ is the index set of $\Gamma$.
Define the function $f$ as:

$$f(x) = \begin{cases} a + 1 & \text{if } x = 0 \\ \uparrow & \text{otherwise.} \end{cases}$$

Then, $f$ is partially computable and does not have $a$ in its range.
Now, extend $f$ to $g$ as

$$g(x) = \begin{cases} a + 1 & \text{if } x = 0 \\ a & \text{otherwise.} \end{cases}$$

Note that $g$ is computable, $a \in \text{range}(g)$, $f \subseteq g$ and if $g = \Phi_b$, then $b \not\in R_a$. Therefore, $R_a$ is not r.e. by the Other Rice’s Theorem.