THEORY OF COMPUTATION
Problem session - 12

Prof. Dan A. Simovici

UMB
1 Problems
Computation in $\mathcal{S}$ Involving Strings

\[
R^+(x, y) = \begin{cases} 
R(x, y) & \text{if } \sim (y | x) \\
y & \text{otherwise,} 
\end{cases}
\]

\[
Q^+(x, y) = \begin{cases} 
\lfloor x/y \rfloor & \text{if } \sim (y | x) \\
\lfloor x/y \rfloor - 1 & \text{otherwise.} 
\end{cases}
\]

If \( u_0 = i_k \cdot n^k + i_{k-1} \cdot n^{k-1} + \cdots + i_1 \cdot n + i_0 \), then

\[i_m = R(u_m, n), u_{m+1} = Q^+(u_m, n),\]

for \( 0 \leq m \leq k \).

Note that the digits are computed from right to left: \( i_0, i_1, \ldots, i_k \).
Computation in $S$ Involving Strings

Useful functions in $S$:

- $\text{CONCAT}_{n}^{(m)}(u_1, \ldots, u_m)$ is the string obtained by placing the strings $u_1, \ldots, u_m$ one after another;
- $\text{RTEND}_{n}(w)$ gives the rightmost symbol of a non-empty word $w$;
- $\text{LTEND}_{n}(w)$ which gives the leftmost symbol of a non-empty word $w$;
- $\text{RTRUNC}_{n}(w)$ which gives the result of removing the rightmost symbol from a given non-empty word;
- $\text{LTRUNC}_{n}(w)$ which gives the result of removing the leftmost symbol from a given non-empty word;
- $\text{DOWNCHANGE}_{n,\ell}$, where $n < \ell$ is used to change base;
- $\text{UPCHANGE}_{n,\ell}$, where $n < \ell$ is used to change base.
All above functions operate on numerical equivalents of words.
Problem 1: Compute $\text{CONCAT}_3^{(2)}(17, 32)$. 
Solution to Problem 1: Begin by determining the strings represented by the numbers 17 and 32 in the base 3:

\[
\begin{align*}
i_0 &= R^+(17, 3) = 2, \quad Q^+(17, 3) = 5, \\
i_1 &= R^+(5, 3) = 2, \quad Q^+(5, 3) = 1, \\
i_2 &= R^+(1, 3) = 1
\end{align*}
\]

\[
\begin{align*}
i_0 &= R^+(32, 3) = 2, \quad Q^+(32, 3) = 10, \\
i_1 &= R^+(10, 3) = 1, \quad Q^+(10, 3) = 3, \\
i_2 &= R^+(3, 3) = 3
\end{align*}
\]

Thus, the strings are \textbf{122} and \textbf{312}, their concatenation is \textbf{122312}, which has the numerical equivalent 491. Thus, \(\text{CONCAT}^{(2)}_3(17, 32) = 491\).
Problem 2: Show that the function \( f(x) \) whose value is the string \( y \) formed of the symbols occurring in the odd-number places in the input \( x \), that is, \( f(a_1 a_2 a_3 \cdots a_m) = a_1 a_3 \cdots \) is computable in \( S \). Assume that \( f(0) = 0 \).
Solution to Problem 2:

\[
[A] \quad \begin{align*}
   & \text{IF } X = 0 \text{ GOTO } E \\
   & Z \leftarrow \text{LTEND}_n(X) \\
   & Y \leftarrow \text{CONCAT}_{(2)}(Y, Z) \\
   & X \leftarrow \text{LTRUNC}_n(X) \\
   & X \leftarrow \text{LTRUNC}_n(X) \\
   & \text{GOTO } A
\end{align*}
\]
Problem 3: Show that the function $\text{UPCHANGE}_{n,\ell}$ is primitive recursive.
Solution to Problem 3: We already know that the functions $\text{UPCHANGE}_{n,\ell}$ and $\text{DOWNCHANGE}_{n,\ell}$ are computable. Now (in Problems 3 and 4) we have to show that they are primitive recursive.

Example: 5 in “base 2” is $5 = 2^1 \cdot 2 + 1$, written as 21. In “base 6” this is $6^1 \cdot 2 + 1 = 13$, so $\text{UPCHANGE}_{2,6}(5) = 13$.

So, $\text{UPCHANGE}_{n,\ell}$ starts with a number, transforms it into a string of digits in “base $n$”, then computes the number in “base” $\ell$.

Begin by noting that $\text{UPCHANGE}_{n,\ell}(0) = 0$. We need to express $\text{UPCHANGE}_{n,\ell}(x + 1)$ through $\text{UPCHANGE}_{n,\ell}(x)$. 
The last digit of a number $x$ in base $n$ is $i_0 = R^+(x, n)$. This allows us to write:

$$ \text{UPCHANGE}_{n, \ell}(x + 1) = \text{UPCHANGE}_{n, \ell}(x) \cdot \ell + R^+(x, n). $$

The equalities

$$ \text{UPCHANGE}_{n, \ell}(0) = 0, $$
$$ \text{UPCHANGE}_{n, \ell}(x + 1) = \text{UPCHANGE}_{n, \ell}(x) \cdot \ell + R^+(x, n). $$

constitute a definition by primitive recursion of the function $\text{UPCHANGE}_{n, \ell}$, which shows that $\text{UPCHANGE}_{n, \ell}$ is indeed primitive recursive.
Problem 4: Show that the function DOWNCHANGE_{n,ℓ} is primitive recursive.

Example: $s_2s_6s_1$ in base 6 is represented by the number

$$2 \cdot 6^2 + 6 \cdot 6^1 + 1 = 109.$$ 

To down change to base 2, we need to cut $s_6$ yielding $s_2s_1$ which in base 2 equals 5. Thus, $\text{DOWNCHANGE}_{2,6}(109) = 5$.

Similarly, $s_6s_2s_6$ in base 6 is represented by the number

$$6 \cdot 6^2 + 2 \cdot 6^1 + 6 = 234.$$ 

To down change to the base 2 we remove the $s_6$ symbols and get $s_2$, which in base 2 equals 2. Thus, $\text{DOWNCHANGE}_{2,6}(234) = 2$. 
Solution for Problem 4: Note that $\text{DOWNCHANGE}_{n,\ell}(0) = 0$. In addition, we can write

$$\text{DOWNCHANGE}_{n,\ell}(x + 1) = \begin{cases} 
\text{DOWNCHANGE}_{n,\ell}(x) & \text{if } R^+(x + 1, \ell) > n, \\
\text{DOWNCHANGE}_{n,\ell}(x) + R^+(x + 1, \ell) & \text{if } R^+(x + 1, \ell) \leq n,
\end{cases}$$
Problem 5: Let $x$ be a number and let $|w|$ be the length of the word $w$ that represents $x$ in the base $n$, where $n \geq 2$. Prove that we have:

$$|w| \leq \lfloor \log_n x \rfloor + 1$$

for all $n \in \mathbb{N}$. 
Solution for Problem 5: Suppose that \( w = s_{i_k} s_{i_{k-1}} \cdots s_1 s_0 \) and
\[
x = i_k \cdot n^k + i_{k-1} \cdot n^{k-1} + \cdots + i_1 \cdot n + i_0.
\]

We have \( |w| = k + 1 \), and
\[
n^k + n^{k-1} + \cdots + n + 1 \leq x \leq n^{k+1} + n^k + \cdots + n.
\]

Note that this implies
\[
\frac{n^{k+1} - 1}{n - 1} \leq x \leq \frac{n^{k+2} - n}{n - 1}.
\]
The last inequality can be rewritten as:

$$\frac{n|w| - 1}{n - 1} \leq x \leq \frac{n|w| + 1 - n}{n - 1},$$

or

$$\log_n \frac{n|w| - 1}{n - 1} \leq \log_n x \leq \log_n \frac{n|w| + 1 - n}{n - 1}.$$
Since $n^{w-1} \leq \frac{n^{w-1}}{n-1}$, it follows that

$$|w| - 1 \leq \log_n x,$$

which implies inequality

$$|w| \leq \lfloor \log_n x \rfloor + 1.$$