THEORY OF COMPUTATION
Problem session - 6

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UMB
1 Problems

2 Solutions
Problem 1: Prove that the functions min and max where

\[
\min(x, y) = \begin{cases} 
  x & \text{if } x \leq y, \\
  y & \text{if } x \geq y,
\end{cases}
\]

and

\[
\max(x, y) = \begin{cases} 
  y & \text{if } x \leq y, \\
  x & \text{if } x \geq y,
\end{cases}
\]

are primitive recursive.
Problem 2: Let

\[ h_1(x, 0) = f_1(x), \]
\[ h_2(x, 0) = f_2(x), \]
\[ h_1(x, t + 1) = g_1(x, h_1(x, t), h_2(x, t)), \]
\[ h_2(x, t + 1) = g_2(x, h_1(x, t), h_2(x, t)). \]

Prove that if \( f_1, f_2, g_1, g_2 \) all belong to some PRC class \( C \), then \( h_1, h_2 \) do also.
Problem 3: Let \( \text{trim} : \mathbb{N} \rightarrow \mathbb{N} \) be the function defined as follows: if \( z = [x_1, \ldots, x_{n-1}, x_n] \), then \( \text{trim}(z) = [x_1, \ldots, x_{n-1}] \). For the special case when \( z = 0 \), we define \( \text{trim}(z) = 0 \). Prove that \( \text{trim} \) is primitive recursive.
Problem 4: The function insert : \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) takes the code of an sorted list of numbers \( x \), and number \( y \), inserts \( y \) in to the list in the correct position and returns the code of the new list. Prove that insert is primitive recursive.
Problem 1: Prove that the functions min and max where

\[
\max(x, y) = \begin{cases} 
  y & \text{if } x \leq y, \\
  x & \text{if } x \geq y, 
\end{cases}
\]

and

\[
\min(x, y) = \begin{cases} 
  x & \text{if } x \leq y, \\
  y & \text{if } x \geq y, 
\end{cases}
\]

are primitive recursive.

**Solution:** Note that

\[
\max(x, y) = \lceil (x + y + |x - y|)/2 \rceil
\]

and

\[
\min(x, y) = \lceil (x + y \div |x - y|)/2 \rceil,
\]

which implies the primitive recursiveness.
Problem 2: Let

\[
\begin{align*}
  h_1(x, 0) &= f_1(x), \\
  h_2(x, 0) &= f_2(x), \\
  h_1(x, t + 1) &= g_1(x, h_1(x, t), h_2(x, t)), \\
  h_2(x, t + 1) &= g_2(x, h_1(x, t), h_2(x, t)).
\end{align*}
\]

Prove that if \( f_1, f_2, g_1, g_2 \) all belong to some PRC class \( C \), then \( h_1, h_2 \) do also.

**Solution:** Define the function \( F(x, t) = \langle h_1(x, t), h_2(x, t) \rangle \).

Clearly,

\[
F(x, 0) = \langle f_1(x), f_2(x) \rangle.
\]

Also,

\[
F(x, t + 1) = \langle h_1(x, t + 1), h_2(x, t + 1) \rangle
= \langle g_1(x, h_1(x, t), h_2(x, t)), g_2(x, h_1(x, t), h_2(x, t)) \rangle
= \langle g_1(x, \ell(F(x, t))), g_2(x, r(F(x, t))) \rangle,
\]

which is a definition be primitive recursion of \( F \). Thus, \( F \in C \).

Since \( h_1(x, t) = \ell(F(x, t)) \) and \( h_2 = r(F(x, t)) \) and \( \ell, r, F \in C \), it follows that \( h_1, h_2 \in C \).
Problem 3: Let \( \text{trim} : \mathbb{N} \rightarrow \mathbb{N} \) be the function defined as follow: if \( z = [x_1, \ldots, x_{n-1}, x_n] \), then \( \text{trim}(z) = [x_1, \ldots, x_{n-1}] \). For the special case when \( z = 0 \), we define \( \text{trim}(0) = 0 \). Prove that \( \text{trim} \) is primitive recursive.

Solution: The primitive recursiveness of \( \text{trim} \) results from the equality

\[
\text{trim}(z) = \min_{u \leq z} \bigwedge_{i=1}^{\text{Lt}(z)-1} [(u)_i = (z)_i].
\]
Problem 4: The function \( \text{insert} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) takes the code of an sorted list of numbers \( x \) and number \( y \), inserts \( y \) in to the list in the correct position and returns the code of the new list. Prove that \( \text{insert} \) is primitive recursive.

Solution: Let \( \mu(x, y) \) be the largest position in \( x \) whose component is smaller than \( y \). We have \( \mu(x, y) = \max_{j \le \text{Lt}(x)} (x)_j < y \). Clearly, \( \mu \) is primitive recursive. Then,

\[
\text{insert}(x, y) = \prod_{i=1}^{\mu(x,y)} p_i^{(x)_i} \cdot p_y^{\mu(x,y)+1} \prod_{k=\mu(x,y)+2}^{p_{(x)_{k-1}}},
\]

which proves that \( \text{insert} \) is primitive recursive.

For example, suppose that \( x = [1, 4, 7, 8, 10] \) and we need to compute \( \text{insert}(x, 6) \). We have \( \mu(x, 6) = 2 \) because \((x)_1 = 1 < 6, (x)_2 = 4 < 6 \) but \((x)_3 = 7 \not< 6 \). Therefore,

\[
\text{insert}(x, 6) = p_1^1 p_2^4 p_3^6 p_4^7 p_5^8 p_6^{10}.
\]