Remainder: Recall that

\[ W_n = \{ x \in \mathbb{N} \mid \Phi(x, n) \downarrow \}. \]

Also, recall that a set \( B \) is recursively enumerable if and only if \( B = W_n \) for some \( n \), that is, if \( B \) is the definition domain of a partial computable function. We also have shown that \( B \) is r.e. if it the range of a partial computable function (or the range of a primitive recursive function, or the range of a recursive function).
Problem 1: Prove that if $A$ is a r.e. set, then $B = \bigcup_{x \in A} W_x$ is also a r.e. set.

Solution for Problem 1: Since $A$ is r.e., there exists $n$ such that $A = W_n$. Then, we have

$$b \in B \iff (\exists x)[x \in A \& b \in W_x]$$
(by the definition of $B$)

$$\iff (\exists x)(\exists n)[x \in W_n \& b \in W_x]$$

$$\iff (\exists x)(\exists n)[\text{HALT}(x, n) \& \text{HALT}(b, x)]$$

$$\iff (\exists x)(\exists n)[(\exists u)\text{STP}^{(1)}(x, n, u) \& (\exists v)\text{STP}^{(1)}(b, x, v)].$$
Encode $t = [x, n, u, v]$. Then, $b \in B$ if and only if

$$(\exists t)[Lt(t) = 4 \& STP^{(1)}((t)_1, (t)_2, (t)_3) \& STP^{(1)}(b, (t)_1, (t)_4)]$$

If $R(b, t) = (Lt(t) = 4) \& STP^{(1)}((t)_1, (t)_2, (t)_3) \& STP^{(1)}(b, (t)_1, (t)_4)$, then $b \in B \iff (\exists t)R(b, t)$. Thus, $B$ is r.e.
Problem 2: Define the set $A$ as

$$A = \{ n \in \mathbb{N} \mid \{0, 1, \ldots, n\} \subseteq W_n \}.$$ 

Prove that $A$ is r.e.

Solution for Problem 2: The definition of $A$ implies:

$$n \in A \iff \{0, 1, \ldots, n\} \subseteq W_n \iff \Phi(0, n) \downarrow \& \Phi(1, n) \downarrow \& \cdots \& \Phi(n, n) \downarrow \iff (\exists t_0)[\text{STP}^{(1)}(0, n, t_0)] \& (\exists t_1)[\text{STP}^{(1)}(1, n, t_1)] \& \cdots \& (\exists t_n)[\text{STP}^{(1)}(n, n, t_n)].$$
Solution cont’d

Define \( t = [t_0, t_1, \ldots, t_n] \). Then,

\[
\begin{align*}
  n \in A & \iff (\exists t)[(Lt(t) = n + 1) \& \text{STP}^{(1)}(0, n, (t)_0) \& \text{STP}^{(1)}(1, n, (t)_1) \& \cdots \& \text{STP}^{(1)}(n, n, (t)_n)] \\
  & \iff (\exists t)[(Lt(t) = n + 1) \& \bigwedge_{i \leq n} \text{STP}^{(1)}(i, n, (t)_i)].
\end{align*}
\]

If \( R(n, t) = (Lt(t) = n) \& \bigwedge_{i \leq n} \text{STP}^{(1)}(i, n, (t)_i) \), then \( n \in A \) then \( n \in A \iff (\exists t)R(n, t) \) so \( A \) is r.e.
Problem 3: Show that if $A$ is an r.e. set, then $B = \bigcup_{x \in A} W_x$ is also a r.e. set.

Solution for Problem 3: Since $A$ is r.e., there exists $n$ such that $A = W_n$. We have:

$$b \in B \iff (\exists x)[x \in A \& b \in W_x] \iff (\exists x)(\exists n)[x \in W_n \& b \in W_x] \iff (\exists x)(\exists n)(\text{HALT}(n, x) \& \text{HALT}(x, b)) \iff (\exists x)(\exists n)(\text{STP}^{(1)}(x, n, u) \& \text{STP}^{(1)}(b, x, v)).$$
Let \( t = [x, n, u, v] \). We have:

\[
b \in B \iff (\exists t)(\text{Lt}(t) = 4 \& \text{STP}(1)((t)_1, (t)_2, (t)_3) \& \text{STP}(1)(b, (t)_1, (t)_4)).
\]

If \( R(b, t) \) is

\[
(\exists t)(\text{Lt}(t) = 4 \& \text{STP}(1)((t)_1, (t)_2, (t)_3) \& \text{STP}(1)(b, (t)_1, (t)_4)),
\]

we have \( b \in B \iff (\exists t)R(b, t) \). Since \( R(b, t) \) is primitive recursive, \( B \) is r.e. by the Projection Theorem.
Problem 4: Let $A, B$ be two proper non-empty subsets of $\mathbb{N}$. Prove that if both $A - B$ and $B - A$ are non-empty recursive sets, then $A \equiv_m B$.

Solution for Problem 3: Since $B$ is non-empty and proper, there is a $b_1 \in B$ and there is $b_0 \notin B$. Define $f$ as

$$f(x) = \begin{cases} 
  b_1 & \text{if } x \in A - B \\
  b_0 & \text{if } x \in B - A \\
  x & \text{otherwise}
\end{cases}$$

Since $A - B$ and $B - A$ are recursive, $x \in A - B$ and $x \in B - A$ are computable. Thus, $f$ is computable. We have

$$x \in A \Rightarrow (x \in A - B) \lor (x \in A \cap B)$$

$$\Rightarrow (f(x) = b_1) \lor (f(x) = x \in A \cap B)$$

$$\Rightarrow f(x) \in B.$$
We also have

\[ x \notin A \implies (x \in B - A) \lor (x \in \overline{A \cup B}) \]
\[ \implies (f(x) = b_0 \notin B) \lor (f(x) = x \notin B) \]
\[ \implies f(x) \notin B. \]

Therefore, \( x \in A \) if and only if \( f(x) \in B \), so \( A \leq_m B \). Similarly, one can prove that \( B \leq_m A \), so \( A \equiv_m B \).
Problem 5: Let

\[ \text{MONOTONE} = \{ x \mid \Phi_x \text{ is total and } (\forall y)[\Phi_x(y) \leq \Phi_x(y + 1)] \} \].

In other words, \( x \in \text{MONOTONE} \) if \( x \) corresponds to a program that halts on every input and computes a monotonic function. Prove that \( \overline{K} \leq_m \text{MONOTONE} \).
Solution for Problem 5: Define the function \( g \) as

\[
g(y, x) = \begin{cases} 
y & \text{if } \sim \text{STP}(x, x, y), \\
\uparrow & \text{if STP}(x, x, y). 
\end{cases}
\]

\( g \) is a *partially computable function*. Let \( z = \#(P) \), where \( P \) is a program that computes \( g \). By the smn theorem we have

\[
g(y, x) = \Phi(y, x, z) = \Phi(y, S_1^1(x, z)) = \Phi(y, f(x)),
\]

where \( f(x) = S_1^1(x, z) \).
Solution for Problem 5 cont’d: Now we can prove that $\overline{K} \leq_m$ MONOTONE. We have

\[
x \in \overline{K} \rightarrow \Phi(x, x) \uparrow
\rightarrow (\forall y)(\sim \text{STP}(x, x, y))
\rightarrow (\forall y)(g(y, x) = y)
\rightarrow (\forall y)(\Phi(y, f(x)) = y)
\rightarrow \Phi_f(x) \text{ is the identity function}
\rightarrow f(x) \in \text{MONOTONE}.
\]
Solution for Problem 5 cont’d: We also have:

\[ x \notin \overline{K} \quad \rightarrow \quad x \in K \]
\[ \rightarrow \quad \Phi(x, x) \downarrow \]
\[ \rightarrow \quad (\exists y)(S TP(x, x, y)) \]
\[ \rightarrow \quad (\exists y)(g(y, x) \uparrow) \]
\[ \rightarrow \quad (\forall y)(\Phi(y, f(x)) \uparrow) \]
\[ \rightarrow \quad f(x) \text{ is not total} \]
\[ \rightarrow \quad f(x) \notin \text{MONOTONE}. \]

Thus, \( \overline{K} \preceq_m \text{MONOTONE} \).