THEORY OF COMPUTATION
Recursively Enumerable Sets - 10 part 1

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UMB
Predicates can be used to define sets.

**Definition**

If \( P(x_1, \ldots, x_n) \) is a predicate, the set \( B \) defined by \( P \) is

\[
B = \{(x_1, \ldots, x_n) \mid P(x_1, \ldots, x_n) = \text{TRUE}\}.
\]

\( P \) is the characteristic predicate of the set \( B \).

The set \( B \) is defined as computable or recursive if its characteristic predicate is computable.

\( B \) is primitive recursive if \( P \) is a primitive recursive predicate.
In other words, $B$ is recursive if we can give a yes/no answer to the question \( x \in B \). This follows from the fact that $P$ is computable.
The set

$$B = \{(x, y) \mid \text{the program } \mathcal{P} \text{ with } \#(\mathcal{P}) = y \text{ halts on } x\}$$

has $\text{HALT}(X, Y)$ as its characteristic predicate. Since $HALT$ is not computable, the set $B$ is not recursive.
Definition

A set $B$ belongs to a class of functions if its characteristic predicate belongs to that set.
Theorem

Let \( C \) be a PRC class. If \( B, C \) belong to \( C \), then so do the sets \( B \cup C, B \cap C \) and \( \overline{B} \).

Proof.

If \( P_B, P_C \) are the characteristic predicates of \( B \) and \( C \), respectively, and \( P_B, P_C \in C \), then the characteristic predicates of \( B \cup C, B \cap C \) and \( \overline{B} \) are \( P_B \lor P_C, P_B \land P_C \), and \( \neg P_B \), respectively, and we saw that they belong to \( C \).
Theorem

Let $C$ be a PRC class, and let $B \subseteq \mathbb{N}^m$, where $m \geq 1$. Then $B \in C$ if and only if the set of numbers

$$B' = \{[x_1, \ldots, x_m] \mid (x_1, \ldots, x_m) \in B\}$$

belongs to $C$.

Proof.

If $P_B(x_1, \ldots, x_m)$ is the characteristic function of $B$, then

$$P_{B'}(x) \Leftrightarrow P_B((x)_1, \ldots, (x)_m) \& \text{Lt}(x) = m,$$

and $P_{B'}$ clearly belongs to $C$ if $P_B \in C$.

On the other hand, $P_B(x_1, \ldots, x_m) \Leftrightarrow P_{B'}([x_1, \ldots, x_n])$, hence $P_{B'} \in C$ implies $P_B \in C$. 

Definition

The set $B \subseteq \mathbb{N}$ is **recursively enumerable** if there is a partially computable function $g(x)$ such that

$$B = \{x \in \mathbb{N} \mid g(x) \downarrow\}.$$  

The term recursively enumerable is abbreviated as r.e.

A set is recursively enumerable when it the domain of a partially computable function. Equivalently, $B$ is r.e. if it is just the set of inputs on which some program $\mathcal{P}$ halts.
If $P$ is an algorithm for testing the membership in $B$, $P$ will provide an *yes* answer for any $x$ in $B$.

If $x \notin B$ the algorithm $P$ will never terminate. This is why $P$ is also called a *semidecision procedure* for $B$. 
Recursive Enumerable Sets

Recursive Sets

\[
\begin{cases}
1 & \text{if } x \in B \\
? & \text{otherwise}
\end{cases}
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Recursive Enumerable Sets

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\[
\begin{cases}
1 & \text{if } x \in B \\
0 & \text{if } x \notin B
\end{cases}
\]
Theorem

If $B$ is a recursive set, then $B$ is r.e.

Proof.

Since $B$ is recursive, the predicate $x \in B$ is computable, so we can write the program $P$:

$[A] \quad \text{IF } (X \in B) \text{ GOTO } A$

If $h(x)$ is computed by this program then

$B = \{x \in \mathbb{N} \mid h(x) \downarrow\}$.  

\[ \square \]
Theorem

The set $B$ is recursive if and only if both $B$ and $\overline{B}$ are both r.e.

Proof.

If $B$ is recursive, then so is $\overline{B}$, hence both $B$ and $\overline{B}$ are r.e. Conversely, suppose that $B$ and $\overline{B}$ are both r.e., that is

\[
B = \{ x \in \mathbb{N} \mid g(x) \downarrow \},
\]
\[
\overline{B} = \{ x \in \mathbb{N} \mid h(x) \downarrow \},
\]

where $g$ and $h$ are both partially computable.
Proof cont’d

Proof.

Let $g$ be the function computed by program $P$ and $h$ be the function computed by program $Q$, where $\#(P) = p$ and $\#(Q) = q$. The next program computes the characteristic function of $B$:

\[
\begin{align*}
[A] & \quad \text{IF STP}^{(1)}(X, p, T) \ \text{GOTO C} \\
     & \quad \text{IF STP}^{(1)}(X, q, T) \ \text{GOTO E} \\
     & \quad T \leftarrow T + 1 \\
     & \quad \text{GOTO A} \\
[C] & \quad Y \leftarrow 1
\end{align*}
\]
The technique used in the previous proof is known as dovetailing. It combines the algorithms for computing $g$ and $h$ by running the two algorithms for longer and longer times until one of them terminates.
Theorem

If $B$ and $C$ are r.e. sets, then so are $B \cup C$ and $B \cap C$.

Proof.

Let

$$B = \{ x \in \mathbb{N} \mid g(x) \downarrow \} \text{ and } C = \{ x \in \mathbb{N} \mid h(x) \downarrow \},$$

where $g$ and $h$ are partially computable. Let $f$ be computed by

$$Y \leftarrow g(X)$$
$$Y \leftarrow h(X)$$

Note that $f(x) \downarrow$ if and only if $g(x) \downarrow$ and $h(x) \downarrow$. Hence $B \cap C = \{ x \in \mathbb{N} \mid f(x) \downarrow \}$, so $B \cap C$ is r.e.
Proof cont’d

Proof.

For $B \cup C$ we use dovetailing again. Let $g$ be the function computed by program $P$ and $h$ be the function computed by program $Q$, where $\#(P) = p$ and $\#(Q) = q$. Let $k(x)$ be computed by

\[
\begin{align*}
[A] & \quad \text{IF } \text{STP}^{(1)}(X, p, T) \text{ GOTO } E \\
& \quad \text{IF } \text{STP}^{(1)}(X, q, T) \text{ GOTO } E \\
& \quad T \leftarrow T + 1 \\
& \quad \text{GOTO } A
\end{align*}
\]

Thus, $k(x) \downarrow$ just when either $g(x) \downarrow$ or $h(x) \downarrow$, that is $B \cup C = \{x \in \mathbb{N} \mid k(x) \downarrow\}$. \qed
The definition domain of $\Phi_n(x)$ is the set denoted as $W_n$. Equivalently,

$$W_n = \{ x \in \mathbb{N} \mid \Phi(x, n) \downarrow \}.$$

Theorem

Enumeration Theorem: A set $B$ is r.e. if and only if there is an $n$ for which $B = W_n$.

Proof.

This follows immediately from the definition of $\Phi(x, n)$.

The theorem gets its name from the fact that

$$W_0, W_1, \ldots, W_n, \ldots$$

is an enumeration of all r.e. sets.