Finite Automata and Regular Languages  
(part III)

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1 Transition Systems
Nondeterministic finite automata can be further generalized by allowing transitions between states without reading any input symbol.

**Definition**

A transition system (ts) is a 5-tuple $\mathcal{T} = (A, Q, \theta, Q_0, F)$, where $A$, $Q$, and $F$ are as in a finite automaton, $\theta$ is a finite relation, $\theta \subseteq Q \times A^* \times Q$, called the **transition relation of $\mathcal{T}$**, and $Q_0$ is a nonempty subset of $Q$ called the **set of initial states**, and $F$ is the set of **final states**.

A **transition** in $\mathcal{T}$ is a triple $(q, x, q') \in \theta$. We refer to transitions of the form $(q, \lambda, q')$ as **null transitions**.
Transition systems are conveniently represented by labeled directed multigraphs. Namely, if $\mathcal{T} = (A, Q, \theta, Q_0, F)$ is a transition system, then its graph is a labeled directed multigraph $G(\mathcal{T})$.

- $G(\mathcal{T})$ has $Q$ as its set of vertices;
- each directed edge $e$ from $q$ to $q'$ labelled $x$ corresponds to a triple $(q, x, q') \in \theta$, and every such triple is represented by an edge in $G$.

Unlike the graph of a dfa or an ndfa, the edges of the directed graph of a transition system can be labelled by words, including the null word.
Example

The graph of the transition system

\[ T_1 = (\{a, b, c\}, \{q_0, q_1, q_2, q_3\}, \theta, \{q_0\}, \{q_3\}) \]

where \( \theta \) is given by

\[ \theta = \{(q_0, ab, q_1), (q_0, \lambda, q_2), (q_1, bc, q_2), (q_1, \lambda, q_3), (q_2, c, q_3)\} \]

is shown below:
Extending the transition relation

As we did with the transition functions of dfas and ndfas, we wish to extend the transition relation $\theta$ of a transition system $\mathcal{T}$ to the set $Q \times A^* \times Q$. The extension $\theta^* \subseteq Q \times A^* \times Q$ is given next.

1. For every $q \in Q$ define $(q, \lambda, q) \in \theta^*$.
2. Every triple $(q, x, q') \in \theta$ belongs to $\theta^*$.
3. If $(q, x, q'), (q', y, q'') \in \theta^*$, then $(q, xy, q'') \in \theta^*$.

Note that $(q, w, q') \in \theta^*$ if and only if there is a path in $G(\mathcal{T})$ that begins with $q$ and ends with $q'$ such that the concatenated labels of the directed edges of this path form the word $w$. 
If $T$ is the above transition system, then $(q_0, abbcc, q_3) \in \theta^*$ because $(q_0, ab, q_1), (q_1, bc, q_2), (q_2, c, q_3) \in \theta$. Similarly, $(q_0, c, q_3) \in \theta$ because $(q_0, \lambda, q_2), (q_2, c, q_3) \in \theta$. 
**Definition**

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. The **language accepted by** $\mathcal{T}$ is

$$L(\mathcal{T}) = \{ x \in A^* \mid (q_0, x, q) \in \theta^* \text{ for some } q_0 \in Q_0, q \in F \}.$$  

Thus, a word $x$ belongs to $L(\mathcal{T})$ if there is a path in $G(\mathcal{T})$ that begins in an initial state $q_0 \in Q_0$, labeled by $x$, such that the path ends in one of the states of $F$, the set of final states.
Transition systems generalize ndfas

If \( M = (A, Q, \delta, q_0, F) \) is a nondeterministic automaton, define the transition system \( T_M = (A, Q, \theta, \{q_0\}, F) \), where

\[
\theta = \{(q, a, q') \mid q, q' \in Q, a \in A \text{ and } q' \in \delta(q, a)\}.
\]

It can be shown by induction on \(|x|\) that \((q, x, q') \in \theta^*\) if and only if \(q' \in \delta^*(q, x)\). This implies that \( L(T_M) = L(M) \). Therefore, every regular language can be accepted by a transition system. Furthermore, any language that can be accepted by a transition system is regular.
Lemma

For every transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$ there exists a transition system $\mathcal{T}' = (A, Q', \theta', Q_0, F)$ such that $(q, x, q_1) \in \theta'$ implies $|x| \leq 1$ and $L(\mathcal{T}') = L(\mathcal{T})$. 
Define the relation $\theta'$ and the set $Q'$ as follows:

1. Every state $q \in Q$ also belongs to $Q'$.
2. Every triple $(q, x, q') \in \theta$ such that $|x| \leq 1$ also belongs to $\theta'$.
3. If $t = (q, x, q') \in \theta$ such that $x = a_0 \ldots a_{n-1}$ and $n \geq 2$, add $n - 1$ new states $q^t_0, \ldots q^t_{n-2}$ to $Q'$ and the triples

$$(q, a_0, q^t_0), (q^t_0, a_1, q^t_1), \ldots, (q^t_{n-2}, a_{n-1}, q')$$


to $\theta'$.

The ts $\mathcal{T}'$ clearly satisfies the conditions of the lemma, since $(q, x, q') \in \theta^*$ if and only if $(q, x, q') \in \theta'^*$. 
**Theorem**

For every transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$ there exists a deterministic finite automaton $\mathcal{M}$ such that $L(\mathcal{T}) = L(\mathcal{M})$.

**Proof.**

By the previous Lemma we can assume that $(q, x, q') \in \theta$ implies $|x| \leq 1$. Define the deterministic finite automaton $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q'_0, F')$, where the initial state of $\mathcal{M}$ is

$$Q'_0 = \{ q \in Q \mid (q_0, \lambda, q) \in \theta^* \text{ for some } q_0 \in Q_0 \} ,$$

the set of final states is $F' = \{ S \mid S \subseteq Q, S \cap F \neq \emptyset \}$, and the function $\Delta$ is defined by

$$\Delta(S, a) = \{ q' \in Q \mid (q, a, q') \in \theta^* \text{ for some } q \in S \} ,$$

for every $S \subseteq Q$ and $a \in A$. \qed
Proof (cont’d)

It is not difficult to verify, by induction on $|x|$, that

$$\Delta^*(Q_0', x) = \{ q' \in Q \mid (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0 \},$$

for $x \in A^*$. For the basis case, $|x| = 0$, so the above equality becomes

$$Q_0' = \{ q' \in Q \mid (q_0, \lambda, q') \in \theta^* \text{ for some } q_0 \in Q_0 \},$$

which holds by the definition of $Q_0'$. 
Proof (cont’d)

Suppose that the equality holds for words of length $n$, and let $y$ be a word of length $n + 1$. We can write $y = xa$, so

$$\Delta^*(Q'_0, y) = \Delta^*(Q'_0, xa)$$
$$= \Delta(\Delta^*(Q'_0, x), a)$$
$$= \Delta(\{ q' \in Q \mid (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0 \}, a)$$
(by the inductive hypothesis)
$$= \{ r \in Q \mid (q', a, r) \in \theta^*, \text{ for some } q' \text{ such that } (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0 \}$$
(by the definition of $\Delta$)
$$= \{ r \in Q \mid (q_0, xa, r) \in \theta^* \text{ for some } q_0 \in Q_0 \}$$
(by the definition of $\theta^*$)
$$= \{ r \in Q \mid (q_0, y, r) \in \theta^* \text{ for some } q_0 \in Q_0 \},$$
which concludes our inductive argument.
Proof (cont’d)

From this it follows that \( L(\mathcal{M}) = L(\mathcal{T}) \). By definition, \( x \in L(\mathcal{M}) \) if and only if \( \Delta^*(Q_0', x) \in F' \). This is equivalent to \( \Delta^*(Q_0', x) \cap F \neq \emptyset \). This is equivalent to the existence of a state \( q' \in F' \) such that \( (q_0, x, q') \in \theta^* \) for some \( q_0 \in Q_0 \), and this is equivalent to \( x \in L(\mathcal{T}) \).
Corollary

The class of languages that are accepted by transition systems is the class \( \mathcal{R} \) of regular languages.
Definition

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. The $\lambda$-closure is the mapping $K_\mathcal{T} : \mathcal{P}(Q) \rightarrow \mathcal{P}(Q)$ given by

$$K_\mathcal{T}(S) = \{ q \in Q \mid (s, \lambda, q) \in \theta^* \text{ for some } s \in S \}.$$ 

The set $K_\mathcal{T}(S)$ comprises the states in $S$ plus all the states that can be reached from a state in $S$ using a series of $\lambda$-transitions.
Theorem

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. The $\lambda$-closure of $\mathcal{T}$ has the following properties.

1. $S \subseteq K_{\mathcal{T}}(S)$;
2. $S \subseteq S'$ implies $K_{\mathcal{T}}(S) \subseteq K_{\mathcal{T}}(S')$;
3. $K_{\mathcal{T}}(K_{\mathcal{T}}(S)) = K_{\mathcal{T}}(S),$

for every $S, S' \in \mathcal{P}(Q)$. 
Since \((s, \lambda, s) \in \theta^*\) it is immediate that \(S \subseteq K_J(S)\) for every \(S \in \mathcal{P}(Q)\). The second part of the theorem is a direct consequence of the definition of \(K_J\).

Note that Parts (i) and (ii) imply \(K_J(S) \subseteq K_J(K_J(S))\). Let \(q \in K_J(K_J(S))\). There is a state \(s \in S\) and a state \(r \in K_J(S)\) such that \((s, \lambda, r) \in \theta^*\) and \((r, \lambda, q) \in \theta^*\). By the definition of \(\theta^*\) we obtain \((s, \lambda, q) \in \theta^*\), so \(q \in K_J(S)\). This implies \(K_J(K_J(S)) \subseteq K_J(S)\), which gives the last part of the theorem.
**Definition**

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. A $K_{\mathcal{T}}$-closed subset of $Q$ is a set $S$ such that $S \subseteq Q$ and $K_{\mathcal{T}}(S) = S$. 
Example

Consider the transition system

\[ \mathcal{T} = (\{a, b\}, \{q_0, q_1, q_2, q_3\}, \theta, \{q_0\}, \{q_2, q_3\}) \]

whose graph is shown
The closed subsets of $Q$ are $\emptyset$, $\{q_3\}$, $\{q_1, q_2, q_3\}$, and $Q$ itself.
Theorem

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system and let $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q'_0, F')$ be the dfa constructed in earlier. The set $\Delta(S, a)$ is a $K_\mathcal{T}$-closed set of states for every subset $S$ of $Q$ and $a \in A$. 
Proof

To prove the theorem it suffices to show that $K_J(\Delta(S, a)) \subseteq \Delta(S, a)$. Let $p \in K_J(\Delta(S, a))$. There is $p_1 \in \Delta(S, a)$ such that $(p_1, \lambda, p) \in \theta^*$. The definition of $\Delta(S, a)$ implies the existence of $q \in S$ such that $(q, a, p_1) \in \theta^*$. Thus, $(q, a, p) \in \theta^*$, so $p \in \Delta(q, a)$. 
Corollary

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system, and let $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q'_0, F')$ be the constructed dfa. The accessible states of the dfa $\mathcal{M}$ are $K_\mathcal{T}$-closed subsets of $Q$.

Proof.

The initial state $Q'_0$ of $\mathcal{M}$ is obviously closed. If $Q'$ is an accessible state of $\mathcal{M}$, then $Q' = \Delta(S, a)$ for some $S \subseteq Q$. Therefore $Q'$ is closed.
Theorem

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system, and let $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q'_0, F')$ be the dfa constructed earlier. Then, $\Delta(S, a) = K_{\mathcal{T}}(\{q \in Q \mid (s, a, q) \in \theta \text{ for some } s \in S\})$, where $S$ is an accessible state of $\mathcal{M}$. 
Proof

Let $s \in S$. Note that if $(s, a, q) \in \theta$ and $(q, \lambda, q_1) \in \theta^*$, then by the definition of $\theta^*$ we have $(s, a, q_1) \in \theta^*$. Therefore,

$$K_I(\{q \in Q \mid (s, a, q) \in \theta\} \subseteq \Delta(S, a).$$

To prove the converse inclusion, $\Delta(S, a) \subseteq K_I(\{q \in Q \mid (s, a, q) \in \theta\}$, let $(s, a, q_1) \in \theta^*$. Then, there is a path in $G(I)$ that begins with $s$ and ends with $q_1$ such that the concatenated labels of the directed edges of this path form the word $a$. This implies the existence of the states $p, p' \in Q$ such that $(s, \lambda, p) \in \theta^*$, $(p, a, p_1) \in \theta$, and $(p_1, \lambda, q_1) \in \theta^*$. Since $S$ is $K_I$-closed it follows that $p \in S$ and this gives the desired conclusion.
An Algorithm for Constructing a dfa corresponding to a ts

**Input:** A transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$.

**Output:** An accessible dfa $\mathcal{M}_1$ such that $L(\mathcal{M}_1) = L(\mathcal{T})$.

**Method:** Compute the increasing sequence of collections of subsets of $Q$, $Q_0, \ldots, Q_i, \ldots$, where

\[
Q_0 = \{ Q'_0 \}
\]
\[
Q_{i+1} = Q_i \cup \{ U \in \mathcal{P}(Q) \mid U = \Delta(S, a) \text{ for some } S \in Q_i \text{ and } a \in A \}.
\]
the computation of $U = \Delta(S, a)$ can be done by computing first the set $W = \{ q \in Q \mid (s, a, q) \in \theta \text{ for some } s \in S \}$ and then $U = K_T(W)$. Stop when $Q_{i+1} = Q_i$. The set $Q_i$ is the set of accessible states of $\mathcal{M}$. Output $\mathcal{M}' = ACC(\mathcal{M})$, the accessible component of $\mathcal{M}$. 
For the transition system
the transition system is:

\[
\begin{align*}
Q &= \{ q_1, q_2, q_3 \} \\
\emptyset &= \{ q_3 \}
\end{align*}
\]