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5 Duality
Let $E$ be a subset of $\mathbb{R}$.
A function $f : E \rightarrow \mathbb{R}$ has a maximum $M$ on $E$ if there exists $x_0 \in E$ such that $f(x_0) = M$ and $f(x_1) \leq M$ for every $x_1 \in E$. The element $x_0$ is a maximizer of $f$ on $E$.
Similarly, $f : E \rightarrow \mathbb{R}$ has a minimum $m$ on $E$ if there exists $x_0 \in E$ such that $f(x_0) = m$ and $f(x_1) \geq m$ for every $x_1 \in E$. The element $x_0$ is a minimizer of $f$ on $E$. 
If \( f : [a, b] \rightarrow \mathbb{R} \) and \( f \) is continuous, then \( f \) has a global maximum \( M \) and a global minimum \( m \) on \([a,b]\).

If \( f \) has a derivative on \([a, b]\), and \( f'(x_0) = 0 \), then \( x_0 \) is a critical point of \( f \).

A local extremum (minimum or maximum) can occur only at a critical point \( x_0 \). If \( f''(x_0) < 0 \), the critical point provides a local maximum; if \( f''(x_0) > 0 \) the critical point provides a local minimum.
The $\nabla f$ notation

(read “nabla f”).

Let $f : X \rightarrow \mathbb{R}$, where $X \subseteq \mathbb{R}^n$, and let $z \in X$. The gradient of $f$ in $z$ is the vector

$$
(\nabla f)(z) = \begin{pmatrix}
\frac{\partial f}{\partial x_1}(z) \\
\vdots \\
\frac{\partial f}{\partial x_n}(z)
\end{pmatrix} \in \mathbb{R}^n.
$$
Example

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be the function $f(x) = x_1^2 + \cdots + x_n^2$; in other words, $f(x) = \|x\|^2$.

We have

$$\frac{\partial f}{\partial x_1} = 2x_1, \ldots, \frac{\partial f}{\partial x_n} = 2x_n.$$

Therefore, $(\nabla f)(x) = 2x$. 

Example

Let \( b_j \in \mathbb{R}^n \) and \( c_j \in \mathbb{R} \) for \( 1 \leq j \leq n \), and let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be the function

\[
f(x) = \sum_{j=1}^{n} (b'_j x - c_j)^2.
\]

We have \( \frac{\partial f}{\partial x_i}(x) = \sum_{j=1}^{n} 2b_{ij}(b'_j x - c_j) \), where \( b_j = (b_{1j} \cdots b_{nj}) \) for \( 1 \leq j \leq n \). Thus, we obtain:

\[
(\nabla f)(x) = 2 \begin{pmatrix}
\sum_{j=1}^{n} 2b_{1j}(b'_j x - c_j) \\
\vdots \\
\sum_{j=1}^{n} 2b_{nj}(b'_j x - c_j)
\end{pmatrix} = 2(B'x - c')B = 2B'xB - 2c'B,
\]

where \( B = (b_1 \cdots b_n) \in \mathbb{R}^{n \times n} \).
The matrix-valued function \( H_f : \mathbb{R}^k \rightarrow \mathbb{R}^{k \times k} \) defined by

\[
H_f(x) = \left( \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}} \right)
\]

is the \textit{Hessian matrix} of \( f \).
Example

For the function \( f(x) = x_1^2 + \cdots + x_n^2 \) discussed on Slide 6 we have

\[
H_f(x) = \begin{pmatrix}
2 & 0 & \cdots & 0 \\
0 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 2 \\
\end{pmatrix}.
\]
Definition

Let $X$ be a open subset in $\mathbb{R}^n$ and let $f : X \longrightarrow \mathbb{R}$ be a function. The point $x_0 \in X$ is a **local minimum** for $f$ if there exists $\delta > 0$ such that $B(x_0, \delta) \subseteq X$ and $f(x_0) \leq f(x)$ for every $x \in B(x_0, \delta)$.

The point $x_0$ is a **strict local minimum** if $f(x_0) < f(x)$ for every $x \in B(x_0, \delta) - \{x_0\}$. 
A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is **positive semidefinite** if $x'Ax \geq 0$ for all $x \in \mathbb{R}^n$.

A is **positive definite** if $x'Ax > 0$ for all $x \in \mathbb{R}^n - \{0_n\}$.
Example

The symmetric real matrix

\[ A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \]

is positive definite if and only if \( a > 0 \) and \( b^2 - ac < 0 \). Indeed, we have \( x'Ax > 0 \) for every \( x \in \mathbb{R}^2 - \{0\} \) if and only if \( ax_1^2 + 2bx_1x_2 + cx_2^2 > 0 \), where \( x' = (x_1 \ x_2) \); elementary algebra considerations lead to \( a > 0 \) and \( b^2 - ac < 0 \).
Is the matrix \( A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \)
positive definite?
No, because \((x_1 \ x_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + 4x_1x_2 + x_2^2\) can be made negative with \(x_1 = 1\) and \(x_2 = -1\).
Theorem

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if and only if all its leading principal minors are positive.

The leading minors of the previous matrix are 1 and $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$. 
Theorem

Let $f : B(x_0, r) \rightarrow \mathbb{R}$ be a function that belongs to the class $C^2(B(x_0, r))$, where $B(x_0, r) \subseteq \mathbb{R}^k$ and $x_0$ is a critical point for $f$. If the Hessian matrix $H_f(x_0)$ is positive semidefinite, then $x_0$ is a local minimum for $f$; if $H_f(x_0)$ is negative semidefinite, then $x_0$ is a local maximum for $f$. 
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function in $C^2(B(x_0, r))$. The Hessian matrix in $x_0$ is

$$H_f(x_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} (x_0).$$

Let $a_{11} = \frac{\partial^2 f}{\partial x_1^2}(x_0)$, $a_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0)$, and $a_{22} = \frac{\partial^2 f}{\partial x_2^2}(x_0)$. Note that

$$h' H_f(x_0) h = a_{11} h_1^2 + 2a_{12} h_1 h_2 + a_{22} h_2^2 = h_2^2 (a_{11} \xi^2 + 2a_{12} \xi + a_{22}),$$

where $\xi = \frac{h_1}{h_2}$. 
For a critical point $x_0$ we have:

1. $h^T H_f(x_0) h \geq 0$ for every $h$ if $a_{11} > 0$ and $a_{12}^2 - a_{11}a_{22} < 0$; in this case, $H_f(x_0)$ is positive semidefinite and $x_0$ is a local minimum;

2. $h^T H_f(x_0) h \leq 0$ for every $h$ if $a_{11} < 0$ and $a_{12}^2 - a_{11}a_{22} < 0$; in this case, $H_f(x_0)$ is negative semidefinite and $x_0$ is a local maximum;

3. if $a_{12}^2 - a_{11}a_{22} \geq 0$; in this case, $H_f(x_0)$ is neither positive nor negative definite, so $x_0$ is a saddle point.

Note that in the first two previous cases we have $a_{12}^2 < a_{11}a_{22}$, so $a_{11}$ and $a_{22}$ have the same sign.
Example

Let $a_1, \ldots, a_m$ be $m$ points in $\mathbb{R}^n$. The function $f(x) = \sum_{i=1}^{m} \| x - a_i \|^2$ gives the sum of squares of the distances between $x$ and the points $a_1, \ldots, a_m$. We will prove that this sum has a global minimum obtained when $x$ is the barycenter of the set $\{a_1, \ldots, a_m\}$. 
Example (cont’d)

We have

\[
f(x) = m \lVert x \rVert^2 - 2 \sum_{i=1}^{m} a'_i x + \sum_{i=1}^{m} \lVert a_i \rVert^2
\]

\[
= m(x_1^2 + \cdots + x_n^2) - 2 \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_j + \sum_{i=1}^{m} \lVert a_i \rVert^2,
\]

which implies

\[
\frac{\partial f}{\partial x_j} = 2mx_j - 2 \sum_{i=1}^{m} a_{ij}
\]

for \(1 \leq j \leq n\). Thus, there exists only one critical point given by

\[
x_j = \frac{1}{m} \sum_{i=1}^{m} a_{ij}
\]

for \(1 \leq j \leq n\).
The Hessian matrix $H_f = 2mI_n$ is positive definite, so the critical point is a local minimum and, in view of convexity of $f$, the global minimum. This point is the barycenter of the set $\{a_1, \ldots, a_m\}$. 
Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), \( c : \mathbb{R}^n \rightarrow \mathbb{R}^m \), and \( d : \mathbb{R}^n \rightarrow \mathbb{R}^p \) be three functions defined on \( \mathbb{R}^n \). A general formulation of a **constrained optimization problem** is:

\[
\text{minimize } f(x), \text{ where } x \in \mathbb{R}^n,
\]

subject to \( c(x) \leq 0_m \), where \( c : \mathbb{R}^n \rightarrow \mathbb{R}^m \),

and \( d(x) = 0_p \), where \( d : \mathbb{R}^n \rightarrow \mathbb{R}^p \).
Here $c$ specifies *inequality constraints* placed on $x$, while $d$ defines *equality constraints*.

The **feasible region** of the constrained optimization problem is the set

$$R_{c,d} = \{ x \in \mathbb{R}^n \mid c(x) \leq 0_m \text{ and } d(x) = 0_p \}.$$ 

If the feasible region $R_{c,d}$ is non-empty and bounded, then, under certain conditions a solution exists. If $R_{c,d} = \emptyset$ we say that the constraints are **inconsistent**.
If only inequality constraints are present (as specified by the function $c$) the feasible region is:

$$R_c = \{ x \in \mathbb{R}^n \mid c(x) \leq 0_m \}. $$
Let $x \in R_c$. The *set of active constraints* at $x$ is

$$\text{ACT}(R_c, c, x) = \{i \in \{1, \ldots, m\} \mid c_i(x) = 0\}.$$ 

If $i \in \text{ACT}(R_c, c, x)$, we say that $c_i$ is an *active constraint* or that $c_i$ is *tight* on $x \in S$; otherwise, that is, if $c_i(x) < 0$, $c_i$ is an *inactive* constraint on $x$. 
Definition

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( c : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be two functions. The minimization problem \( \text{MP}(f, c) \) is:

\[
\text{minimize } f(x), \text{ where } x \in \mathbb{R}^n, \\
\text{subject to } x \in R_c.
\]

If \( x_0 \) exists in \( R_c \) that \( f(x_0) = \min\{f(x) \mid x \in R_c\} \) we refer to \( x_0 \) as a solution of \( \text{MP}(f, c) \).
If $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ we can write

$$h(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_m(x) \end{pmatrix},$$

where $h_j : \mathbb{R}^n \rightarrow \mathbb{R}$ are the \textit{components of $h$ for $1 \leq j \leq m$}. If $h$ is a differentiable function, the function $(Dh)(x)$ is

$$(Dh)(x) = \begin{pmatrix} (\nabla h_1)(x)' \\ \vdots \\ (\nabla h_m)(x)' \end{pmatrix}.$$
Example

Let \( h : \mathbb{R}^2 \to \mathbb{R}^3 \) be given by

\[
h(x) = \begin{pmatrix} x_1 x_2 \\ x_1^2 \\ x_2^2 \end{pmatrix}
\]

Then

\[
(Dh)(x) = \begin{pmatrix} x_2 & x_1 \\ 2x_1 & 0 \\ 0 & 2x_2 \end{pmatrix}.
\]
Theorem

(Existence Theorem of Lagrange Multipliers) Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( h : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be two functions such that:

- \( m < n \),
- \( f \in C^1(\mathbb{R}^n) \),
- \( h \in C^1(\mathbb{R}^n) \), and
- the matrix \((Dh)(x)\) is of full rank, that is, \( \text{rank}((Dh)(x)) = m < n \).

If \( x_0 \) is a regular point of \( h \) and a local extremum of \( f \) subjected to the restriction \( h(x_0) = 0_m \), then \((\nabla f)(x_0)\) is a linear combination of \((\nabla h_1)(x_0), \ldots, (\nabla h_m)(x_0)\).
Example

Suppose that we wish to minimize $f(x) = x_1 + x_2$ subject to the condition

$$h(x) = x_1^2 + x_2^2 = 2.$$  

We have

$$\nabla f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\nabla h(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}.$$
At the local minimum $\mathbf{x}^* = (-1, -1)$ we have $(\nabla f)(\mathbf{x}^*) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $(\nabla h) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, so

$$(\nabla f)(\mathbf{x}^*) + \frac{1}{2}(\nabla h) = 0.$$
To apply the Lagrange multiplier technique the constraint gradients

\[(\nabla h_1)(\mathbf{x}), \ldots, (\nabla h_m)(\mathbf{x})\]

must be linearly independent. In this case, \(\mathbf{x}\) is said to be regular. There may not exist Lagrange multipliers for a local minimum that is not regular.
Example

Consider minimizing the function $f(x) = x_1 + x_2$ subject to the constraints

$$h_1(x) = (x_1 - 1)^2 + x_2^2 - 1 = 0, \quad h_2(x) = (x_1 - 2)^2 + x_2^2 - 4 = 0.$$ 

We have

$$\nabla f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$\nabla h_1(x) = \begin{pmatrix} 2(x_1 - 1) \\ 2x_2 \end{pmatrix}, \quad \nabla h_2(x) = \begin{pmatrix} 2(x_1 - 2) \\ 2x_2 \end{pmatrix}.$$
Example continued

The local minimum is at \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). At that point, we have

\[
(\nabla f)(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
(\nabla h_1)(0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix},
(\nabla h_2)(0) = \begin{pmatrix} -4 \\ 0 \end{pmatrix}.
\]

The gradients \((\nabla h_1)(0), (\nabla h_2)(0)\) are not linearly independent, so \(\mathbf{0}\) is not a regular point and Lagrange's multipliers do not exist.
Example

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^\prime A x$.

**Optimization problem:** minimize $f$ subjected to the restriction $\| x \| = 1$, or equivalently $h(x) = \| x \|^2 - 1 = 0$.

Since $(\nabla f) = 2Ax$ and $(\nabla h)(x) = 2x$ there exists $\lambda$ such that $2Ax_0 = 2\lambda x_0$ for any extremum of $f$ subjected to $\| x_0 \| = 1$. Thus, $x_0$ must be a unit eigenvector of $A$ and $\lambda$ must be an eigenvalue of the same matrix.
The next theorem provides necessary conditions for optimality that include the linear independence of the gradients of the components of the constraint \((\nabla c_i)(x_0)\) for \(i \in \text{ACT}(S, c, x_0)\} \) and ensure that the coefficient of the gradient of the objective function \((\nabla f)(x_0)\) is not null. These conditions are known as the \textit{Karush-Kuhn-Tucker conditions} or the \textit{KKT conditions}. 

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**Lagrange Multipliers**
Theorem

(Karush-Kuhn-Tucker Theorem) Let $S$ be a non-empty open subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Let $x_0$ be a local minimum in $S$ of $f$ subjected to the restriction $c(x_0) \leq 0_m$.

Suppose that $f$ is differentiable in $x_0$, $c_i$ are differentiable in $x_0$ for $i \in \text{ACT}(S, c, x_0)$, and $c_i$ are continuous in $x_0$ for $i \not\in \text{ACT}(S, c, x_0)$.

If $\{(\nabla c_i)(x_0) \mid i \in \text{ACT}(S, c, x_0)\}$ is a linearly independent set, then there exist non-negative numbers $w_i$ for $i \in \text{ACT}(S, c, x_0)$ such that

$$(\nabla f)(x_0) + \sum \{w_i(\nabla c_i)(x_0) \mid i \in \text{ACT}(S, c, x_0)\} = 0_n.$$
Furthermore, if the functions $c_i$ are differentiable in $x_0$ for \( i \not\in \text{ACT}(S, c, x_0) \), then the previous condition can be written as:

1. \((\nabla f)(x_0) + \sum_{i=1}^{m} w_i(\nabla c_i)(x_0) = 0_n;\)
2. \(w'c(x_0) = 0;\)
3. \(w \geq 0_m, \text{ where } w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}.\)
The Primal Problem

Consider the following optimization problem for an object function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a subset $C \subseteq \mathbb{R}^n$, and the constraint functions $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $d : \mathbb{R}^n \rightarrow \mathbb{R}^p$:

\[
\text{minimize } f(x), \text{ where } x \in C,
\]

subject to $c(x) \leq 0_m$

and $d(x) = 0_p$.

We refer to this optimization problem as the \textit{primal problem}. 
Definition

The *Lagrangian* associated to the primal problem is the function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ given by:

$$L(x, u, v) = f(x) + u'c(x) + v'd(x)$$

for $x \in C$, $u \in \mathbb{R}^m$, and $v \in \mathbb{R}^p$.

The component $u_i$ of $u$ is the *Lagrangian multiplier* corresponding to the constraint $c_i(x) \leq 0$; the component $v_j$ of $v$ is the *Lagrangian multiplier* corresponding to the constraint $d_j(x) = 0$. 
**Lemma**

At each feasible \( x \) we have

\[
f(x) = \sup \{ L(x, u, v) \mid u \geq 0_m, v \in \mathbb{R}^p, u_i c_i(x) = 0 \text{ for } 1 \leq i \leq m \}.
\]

**Proof:** at each feasible \( x \) we have \( c_i(x) \leq 0 \) and \( d_i(x) = 0 \), hence

\[
L(x, u, v) = f(x) + u'c(x) + v'd(x) \leq f(x).
\]

The last inequality becomes an equality if \( u_i c_i(x) = 0 \) for \( 1 \leq i \leq m \).
Lemma

The optimal value of the primal problem $f^*$ is

$$f^* = \inf_{x} \sup_{u \geq 0, v} L(x, u, v).$$

Proof: Consider feasible $x$ (designated at $x \in C$). In this case we have

$$f^* = \inf_{x \in C} f(x) = \inf_{x \in C} \sup_{u \geq 0, v} L(x, u, v).$$

When $x$ is not feasible, since $\sup_{u \geq 0, v} L(x, u, v) = \infty$ for any $x \not\in C$, we have

$$\inf_{x \not\in C} \sup_{u \geq 0, v} L(x, u, v) = \infty.$$  Thus, in either case,

$$f^* = \inf_{x} \sup_{u \geq 0, v} L(x, u, v).$$
The dual optimization problem starts with the Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ defined by

$$g(u, v) = \inf_{x \in C} L(x, u, v) \quad (1)$$

and consists of

$$\text{maximize } g(u, v), \text{ where } u \in \mathbb{R}^m \text{ and } v \in \mathbb{R}^p,$$

subject to $u \geq 0_m$. 
Theorem

For every primal problem the Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ defined by Equality (1) is always concave over $\mathbb{R}^m \times \mathbb{R}^p$. 
Proof

For \( u_1, u_2 \in \mathbb{R}^m \) and \( v_1, v_2 \in \mathbb{R}^p \) we have:

\[
g(tu_1 + (1 - t)u_2, tv_1 + (1 - t)v_2) \\
= \inf \{ f(x) + (tu'_1 + (1 - t)u'_2)c(x) + (tv'_1 + (1 - t)v'_2)d(x) \mid x \in S \} \\
= \inf \{ t(f(x) + u'_1c(x) + v'_1d(x)) + (1 - t)f(x) + u'_2c(x) + v'_2d(x) \mid x \in S \} \\
\geq t \inf \{ f(x) + u'_1c(x) + v'_1d \mid x \in S \} \\
+ (1 - t) \inf \{ f(x) + u'_2c(x) + v'_2d(x) \mid x \in S \} \\
= tg(u_1, v_1) + (1 - t)g(u_2, v_2),
\]

which shows that \( g \) is concave.
The concavity of $g$ is significant because a local optimum of $g$ is a global optimum regardless of convexity properties of $f$, $c$ or $d$.

Although the dual function $g$ is not given explicitly, the restrictions of the dual have a simpler form and this may be an advantage in specific cases.

The dual function produces lower bounds for the optimal value of the primal problem.
**Theorem**

(The Weak Duality Theorem) *Suppose that $x_*$ is an optimum of $f$ and $f_* = f(x_*)$, $(u_*, v_*)$ is an optimum for $g$, and $g_* = g(u_*, v_*)$. We have $g_* \leq f_*$. Explained below.*

**Proof:** Since $c(x_*) \leq 0_m$ and $d(x_*) = 0_p$ it follows that

$$L(x_*, u, v) = f(x_*) + u'c(x_*) + v'd(x_*) \leq f_*.$$ 

Therefore, $g(u, v) = \inf_{x \in C} L(x, u, v) \leq f_*$ for all $u$ and $v$. Since $g_*$ is the optimal value of $g$, the last inequality implies $g_* \leq f_*$. 

The inequality of the previous theorem holds when $f_*$ and $g_*$ are finite or infinite. The difference $f_* - g_*$ is the \textit{duality gap} of the primal problem. \textbf{Strong duality} holds when the duality gap is 0.
Note that for the Lagrangian function of the primal problem we can write

\[
\sup_{u \geq 0, v} L(x, u, v) = \sup_{u \geq 0, v} f(x) + u'c(x) + v'd(x)
\]

\[
= \begin{cases} 
  f(x) & \text{if } c(x) \leq 0_m, \\
  \infty & \text{otherwise}
\end{cases}
\]

which implies \( f_* = \inf_{x \in \mathbb{R}^n} \sup_{u \geq 0, v} L(x, u, v) \). By the definition of \( g_* \) we also have

\[
g_* = \sup_{u \geq 0, v} \inf_{x \in \mathbb{R}^n} L(x, u, v).
\]
Thus, the weak duality amounts to the inequality

\[
\sup_{u \geq 0_m, v} \inf_{x \in \mathbb{R}^n} L(x, u, v) \leq \inf_{x \in \mathbb{R}^n} \sup_{u \geq 0_m, v} L(x, u, v),
\]

and the strong duality is equivalent to the equality

\[
\sup_{u \geq 0_m, v} \inf_{x \in \mathbb{R}^n} L(x, u, v) = \inf_{x \in \mathbb{R}^n} \sup_{u \geq 0_m, v} L(x, u, v).
\]
Example

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be the linear function \( f(x) = a'x \), \( A \in \mathbb{R}^{p \times n} \), and \( b \in \mathbb{R}^p \). Consider the primal problem:

\[
\text{minimize } a'x, \text{ where } x \in \mathbb{R}^n, \\
\text{subject to } x \geq 0_n \text{ and } \\
A x - b = 0_p.
\]

The constraint functions are \( c(x) = -x \) and \( d(x) = A x - b \) and the Lagrangian \( L \) is

\[
L(x, u, v) = a'x - u'x + v'(A x - b) \\
= -v'b + (a' - u' + v'A)x.
\]
This yields the dual function
\[ g(u, v) = -v^\prime b + \inf_{x \in \mathbb{R}^n} (a^\prime - u^\prime + v^\prime A)x. \]

Unless \( a^\prime - u^\prime + v^\prime A = 0^\prime_n \) we have \( g(u, v) = -\infty \). Therefore, we have
\[
g(u, v) = \begin{cases} 
-v^\prime b & \text{if } a - u + A^\prime v = 0_n, \\
-\infty & \text{otherwise}.
\end{cases}
\]

Thus, the dual problem is
\[
\text{maximize } g(u, v), \quad \text{subject to } u \geq 0_m.
\]
An equivalent of the dual problem is

\[
\text{maximize } -v^T b,
\]

\[
\text{subject to } a - u + A'v = 0_n
\]

\[
\text{and } u \geq 0_m.
\]

In turn, this problem is equivalent to:

\[
\text{maximize } -v^T b,
\]

\[
\text{subject to } a + A'v \geq 0_n.
\]
Example

The following optimization problem

$$\text{minimize } \frac{1}{2} x' Q x - r' x,$$

where $x \in \mathbb{R}^n$,

subject to $A x \geq b$,

where $Q \in \mathbb{R}^{n \times n}$ is a positive definite matrix, $r \in \mathbb{R}^n$, $A \in \mathbb{R}^{p \times n}$, and $b \in \mathbb{R}^p$ is known as a quadratic optimization problem.
The Lagrangian $L$ is

$$L(x, u) = \frac{1}{2} x' Q x - r' x + u' (A x - b) = \frac{1}{2} x' Q x + (u' A - r') x - u' b$$

and the dual function is $g(u) = \inf_{x \in \mathbb{R}^n} L(x, u)$ subject to $u \geq 0_m$. Since $x$ is unconstrained in the definition of $g$, the minimum is attained when we have the equalities

$$\frac{\partial}{\partial x_i} \left( \frac{1}{2} x' Q x + (u' A - r') x - u' b \right) = 0$$

for $1 \leq i \leq n$, which amount to $x = Q^{-1}(r - A u)$. The dual optimization function is: $g(u) = -\frac{1}{2} u' P u - u' d - \frac{1}{2} r' Q r$ subject to $u \geq 0_p$, where $P = A Q^{-1} A'$, $d = b - A Q^{-1} r$. This shows that the dual problem of this quadratic optimization problem is itself a quadratic optimization problem.
Example

Let $a_1, \ldots, a_m \in \mathbb{R}^n$. We seek to determine a closed sphere $B[x, r]$ of minimal radius that includes all points $a_i$ for $1 \leq i \leq m$. This is the minimum bounding sphere problem, formulated by J. J. Sylvester. This problem amounts to solving the following primal optimization problem:

\[
\text{minimize } r, \text{ where } r \geq 0,
\]

subject to $\| x - a_i \| \leq r$ for $1 \leq i \leq m$. 

An equivalent formulation requires minimizing \( r^2 \) and stating the restrictions as \( \| x - a_i \|^2 - r^2 \leq 0 \) for \( 1 \leq i \leq m \). The Lagrangian of this problem is:

\[
L(r, x, u) = r^2 + \sum_{i=1}^{m} u_i (\| x - a_i \|^2 - r^2)
\]

\[
= r^2 \left( 1 - \sum_{i=1}^{m} u_i \right) + \sum_{i=1}^{m} u_i \| x - a_i \|^2
\]

and the dual function is:

\[
g(u) = \inf_{r \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n} L(r, x, u)
\]

\[
= \inf_{r \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}^n} r^2 \left( 1 - \sum_{i=1}^{m} u_i \right) + \sum_{i=1}^{m} u_i \| x - a_i \|^2
\]
This leads to the following conditions:

\[
\frac{\partial L(r, x, u)}{\partial r} = 2r \left( 1 - \sum_{i=1}^{m} u_i \right) = 0
\]

\[
\frac{\partial L(r, x, u)}{\partial x_p} = 2 \sum_{i=1}^{m} u_i (x - a_i)_p = 0 \text{ for } 1 \leq p \leq n.
\]

The first equality yields \(\sum_{i=1}^{m} u_i = 1\). Therefore, from the second equality we obtain \(x = \sum_{i=1}^{m} u_i a_i\). This shows that for \(x\) is a convex combination of \(a_1, \ldots, a_m\). The dual function is

\[
g(u) = \sum_{i=1}^{m} u_i \left( \sum_{h=1}^{m} u_h a_h - a_i \right) = 0
\]

because \(\sum_{i=1}^{m} u_i = 1\).

Note that the restriction functions \(g_i(x, r) = \| x - a_i \|^2 - r^2 \leq 0\) are not convex.
Example

Consider the primal problem

\[
\text{minimize } x_1^2 + x_2^2, \text{ where } x_1, x_2 \in \mathbb{R}, \\
\text{subject to } x_1 - 1 \geq 0.
\]

It is clear that the minimum of \( f(x) \) is obtained for \( x_1 = 1 \) and \( x_2 = 0 \) and this minimum is 1. The Lagrangian is

\[
L(u) = x_1^2 + x_2^2 + u_1(x_1 - 1)
\]

and the dual function is

\[
g(u) = \inf_x \{ x_1^2 + x_2^2 + u_1(x_1 - 1) \mid x \in \mathbb{R}^2 \} = -\frac{u_1^2}{4}.
\]

Then \( \sup \{ g(u_1) \mid u_1 \geq 0 \} = 0 \) and a gap exists between the minimal value of the primal function and the maximal value of the dual function.
Example

Let $a, b > 0$, $p, q < 0$ and let $r > 0$. Consider the following primal problem:

\[
\text{minimize } f(x) = ax_1^2 + bx_2^2 \\
\text{subject to } px_1 + qx_2 + r \leq 0 \text{ and } x_1 \geq 0, x_2 \geq 0.
\]

The set $C$ is $\{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$. The constraint function is $c(x) = px_1 + qx_2 + r \leq 0$ and the Lagrangian of the primal problem is

\[
L(x, u) = ax_1^2 + bx_2^2 + u(px_1 + qx_2 + r),
\]

where $u$ is a Lagrangian multiplier.
Thus, the dual problem objective function is

\[
g(u) = \inf_{x \in C} L(x, u) \\
= \inf_{x \in C} ax_1^2 + bx_2^2 + u(px_1 + qx_2 + r) \\
= \inf_{x \in C} \{ax_1^2 + upx_1 \mid x_1 \geq 0\} \\
+ \inf_{x \in C} \{bx_2^2 + uqx_2 \mid x_2 \geq 0\} + ur
\]

The infima are achieved when \( x_1 = -\frac{up}{2a} \) and \( x_2 = -\frac{uq}{2b} \) if \( u \geq 0 \) and at \( x = 0_2 \) if \( u < 0 \). Thus,

\[
g(u) = \begin{cases} 
-\left(\frac{p^2}{4a} + \frac{q^2}{4b}\right)u^2 + ru & \text{if } u \geq 0, \\
ru & \text{if } u < 0
\end{cases}
\]

which is a concave function.
The maximum of $g(u)$ is achieved when $u = \frac{2r}{\frac{p^2}{a} + \frac{q^2}{b}}$ and equals

$$r^2 \left( \frac{p^2}{a} + \frac{q^2}{b} \right)$$

Family of Concentric Ellipses; the ellipse that “touches” the line $px_1 + qx_2 + r = 0$ gives the optimum value for $f$. The dotted area is the feasible region.
Note that if \( x \) is located on an ellipse \( ax_1^2 + bx_2^2 - k = 0 \), then \( f(x) = k \). Thus, the minimum of \( f \) is achieved when \( k \) is chosen such that the ellipse is tangent to the line \( px_1 + qx_2 + r = 0 \). In other words, we seek to determine \( k \) such that the tangent of the ellipse at \( x_0 = \left( \begin{array}{c} x_{01} \\ x_{02} \end{array} \right) \) located on the ellipse coincides with the line given by \( px_1 + qx_2 + r = 0 \). The equation of the tangent is

\[
ax_1 x_{01} + bx_2 x_{02} - k = 0.
\]

Therefore, we need to have:

\[
\frac{ax_{01}}{p} = \frac{bx_{02}}{q} = \frac{-k}{r},
\]

hence \( x_{01} = -\frac{kp}{ar} \) and \( x_{02} = -\frac{kq}{br} \). Substituting back these coordinates in the equation of the ellipse yields \( k_1 = 0 \) and \( k_2 = \frac{r^2}{\frac{p^2}{a} + \frac{q^2}{b}} \). In this case no duality gap exists.