Finite Automata and Regular Languages
(Preliminaries)

Prof. Dan A. Simovici

UMB
1. Equivalences

2. Partitions
Definition of Equivalences

Definition

An **equivalence** on a set $S$ is a relation $\rho \subseteq S \times S$ that satisfies the following conditions:

- **Reflexivity**: $(x, x) \in \rho$ for every $x \in S$;
- **Symmetry**: $(x, y) \in \rho$ if and only if $(y, x) \in \rho$;
- **Transitivity**: if $(x, y) \in \rho$ and $(y, z) \in \rho$ then $(x, z) \in \rho$. 
Example

Define the relation $\rho_m$ on $\mathbb{N}$ as

$$\rho_m = \{(p, q) \mid p, q \in \mathbb{N} \text{ and } p - q = km \text{ for some } k \in \mathbb{Z}\}.$$

- $(p, p) \in \rho_m$ because $p - p = 0 \cdot m$, so $\rho_m$ is reflexive;
- if $(p, q) \in \rho_m$, then $p - q = km$, hence $q - p = (-k)m$, which means that $(q, p) \in \rho_m$;
- if $(p, q) \in \rho_m$ and $(q, r) \in \rho_m$ then $p - q = km$ and $q - r = hm$; therefore, $p - r = p - q + q - r = (k + h)m$, so $(p, r) \in \rho_m$, hence $\rho_m$ is transitive.

Thus, $\rho_m$ is an equivalence relation on $\mathbb{N}$. 
Example

Let $L$ be the set of lines in a plane $\Pi$. Define $\ell \parallel \ell'$ if $\ell$ is parallel to $\ell'$. 

- we have $\ell \parallel \ell$, so $\parallel$ is reflexive;
- if $\ell \parallel \ell'$, then $\ell' \parallel \ell$ hence $\parallel$ is symmetric;
- if $\ell \parallel \ell'$ and $\ell' \parallel \ell''$, then $\ell \parallel \ell''$, so parallel is transitive.

Thus, “$\parallel$” is an equivalence relation on the set $L$. 
Definition

Let $S$ be a set and let $\rho$ be an equivalence on $S$. The $\rho$-equivalence class of an element $x$ of $S$ is the set $[x]$ defined by

$$[x] = \{ z \in S \mid (x, z) \in \rho \}.$$ 

The quotient set $S/\rho$ is the set of all equivalence classes defined by $\rho$ on the set $S$. 

Note that:

- we have $x \in [x]$ because $(x, x) \in \rho$, so none of the equivalence classes is empty;
- if $y \in [x]$, then $x \in [y]$ because $\rho$ is symmetric; thus, in this case, $[y] = [x]$.
- if two equivalence classes are distinct, they are disjoint.

Suppose that $[y] \neq [x]$ and there exists $t \in [x] \cap [y]$. Then $(x, t) \in \rho$ and $(y, t) \in \rho$, which means that $(t, y) \in \rho$. By transitivity $(x, y) \in \rho$, which implies $[x] = [y]$, contradicting the initial assumption.
**Definition**

Let $S$ be a non-empty set. A partition on $S$ is a collection of sets

$$\pi = \{B_i \mid B_i \subseteq S \text{ for } i \in I\}$$

such that

- $B_i \neq \emptyset$ for all $i \in I$;
- $B_i \cap B_j = \emptyset$ for $i, j \in I$ and $i \neq j$;
- $\bigcup_{i \in I} B_i = S$. 
Example
If $\rho$ is an equivalence on a set $S$, its set of classes

$$\{[x] \mid x \in S\}$$

is a partition of the set $S$. This shows how we can “move” from equivalences to partitions.
Conversely, if $\pi = \{B_i \mid B_i \subseteq S \text{ for } i \in I\}$ is a partition of $S$, an equivalence $\rho_{\pi}$ is defined by

$$(x, y) \in \rho_{\pi} \text{ if and only if there is a block } B \text{ of } \pi \text{ such that } \{x, y\} \subseteq B.$$
Definition

Let \( \pi \) be a partition of a set \( S \). A subset \( U \) of \( S \) is \( \pi \)-saturated if it equals the union of a collection of blocks of \( \pi \).
Example

Let $S = \{1, 2, \ldots, 9\}$ be a set and let

$$\pi = \{\{1, 2, 7\}, \{4, 6\}, \{3, 5, 8, 9\}\}.$$  

The set $S$ has $2^9 = 512$ subsets. The following 8 subsets of $S$ are $\pi$-saturated:

- $\emptyset$
- $\{1, 2, 7\}, \{4, 6\}, \{3, 5, 8, 9\}$  
- $\{1, 2, 7, 4, 6\}, \{1, 2, 7, 3, 5, 8, 9\}, \{4, 6, 3, 5, 8, 9\}$  
- $\{1, 7, 4, 6, 3, 5, 8, 9\}.$