CS724: Topics in Algorithms
Set-Theoretical Preliminaries
Slide Set 1

Prof. Dan A. Simovici
1. Sets and Set Operations
2. Sequences
3. Closure Systems
4. Inclusion-Exclusion Principle
5. Relations, Equivalence Relations and Partitions
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Set Notations

For a set \( S \) we denote by \( \mathcal{P}(S) \) the set of its subsets. The set of all finite non-empty subsets of \( S \) is denoted by \( \mathcal{F}(S) \).

For a finite set \( S \) the number of elements of \( S \) is denoted by \( |S| \). The empty set is denoted by \( \emptyset \).

We write \( x \in S \) to denote the fact that \( x \) is an element of the set \( S \). The usual symbols are used to denote set-theoretical operations: \( A \cup B \) is the union of the sets \( A \) and \( B \), \( A \cap B \) is the intersection of the sets \( A \) and \( B \), and \( A - B \) is the difference of the sets \( A \) and \( B \).

The symmetric difference of the sets \( A \) and \( B \) is denoted by \( A \oplus B \). We have

\[
A \oplus B = (A - B) \cup (B - A).
\]
Definition

Let $S$ be a set. A sequence of length $n$ on $S$ is a mapping $s : \{1, \ldots, n\} \rightarrow S$. The set of sequences of length $n$ on $S$ is denoted by $\text{Seq}_n(S)$. An ordered pair on $S$ is a sequence of length 2 on $S$; a singleton is a sequence of length 1. If $s$ is a sequence of length $n$ on $S$ and $s(i) = x_i$ for $1 \leq i \leq n$, we write $s = (x_1, \ldots, x_n)$. The elements $x_1, \ldots, x_n$ are the components of $s$.

The length of a sequence $s$ is denoted by $|s|$. 
Example

A sequence of natural numbers of length 6 is $s = (6, 5, 2, 4, 9, 6)$. Note that in a sequence the same element of $S$ may occur on multiple positions.
Counting Sequences

If $S$ is a finite set containing $m$ elements, then there are $m^n$ sequences of length $n$ for any $n \geq 1$. We extend the definition of sequences on $S$ by defining the *null sequence on $S$* as the sequence $\lambda$ that has no components, $\lambda = ()$. Note that there exists exactly one such sequence on $S$ and this is consistent with the fact that $m^0 = 1$ for every $m \geq 1$. The *set of sequences of elements of $S$* is the set

$$
\text{Seq}(S) = \bigcup \{ \text{Seq}_n(S) \mid n \geq 0 \}.
$$
Operations with Sequences

If \( s = (s_1, s_2, \ldots, s_n) \) is a sequence in \( S \), we refer to the sequence \( \tilde{s} = (s_n, \ldots, s_2, s_1) \) as the **reversal** of the sequence \( s \). Clearly \( \tilde{\lambda} = \lambda \).

If \( s = (s_1, \ldots, s_n) \) and \( t = (t_1, \ldots, t_m) \) are two sequences on a set \( S \), their **concatenation** is the sequence \( st = (s_1, \ldots, s_n, t_1, \ldots, t_m) \). For the null sequence we define \( \lambda s = s \lambda = s \) for every \( s \in \text{Seq}(S) \). Note that \( |st| = |s| + |t| \) for all sequences \( s, t \in \text{Seq}(S) \).

Note that sequence concatenation is not a commutative operation in general.
Example

Let \( s = (1, 2, 3) \), \( t = (4, 5) \). We have

\[
st = (1, 2, 3, 4, 5) \quad \text{and} \quad ts = (4, 5, 1, 2, 3),
\]

so \( st \neq ts \).

Sequence concatenation is an associative operation on \( \text{Seq}(S) \), that is

\[
(st)u = s(tu) \quad \text{for every} \quad s, t, u \in \text{Seq}(S).
\]
Operations with Collections of Sets

Let \( \mathcal{C} = \{ S_i \mid i \in I \} \) be a collection of sets. Its union is the set \( U \) defined as

\[
U = \bigcup_{i \in I} S_i.
\]

Note that \( \mathcal{C} \subseteq \mathcal{C}' \) implies \( \bigcup \mathcal{C} \subseteq \bigcup \mathcal{C}' \).

Unlike the union, the intersection is defined only for collections that consist of subsets of a set \( S \).

If \( \mathcal{C} \) is a collection of subsets of \( S \), that is, if \( \mathcal{C} \subseteq \mathcal{P}(S) \), the intersection of \( \mathcal{C} \) is the set of all elements of \( S \) that belong to every set of \( \mathcal{C} \). The intersection of \( \mathcal{C} \) is denoted by \( \bigcap \mathcal{C} \).
If $C$ and $C'$ are two collections of subsets of a set $S$ and $C \subseteq C'$, then $\bigcap C' \subseteq \bigcap C$. If $\emptyset$ is the empty collection of subsets of $S$, we define $\bigcap \emptyset = S$. 
A closure system on the set $S$ is a collection $\mathcal{K}$ of subsets of $S$ such that for every collection of subsets $\mathcal{C}$ such that $\mathcal{C} \subseteq \mathcal{K}$ we have $\bigcap \mathcal{C} \in \mathcal{K}$.

Note that if $\mathcal{K}$ is a closure system on a set $S$, then $S \in \mathcal{K}$ because $S$ is the intersection of the empty collection of subsets of $\mathcal{K}$.
Definition

Let $\mathcal{K}$ be a closure system on a set $S$ and let $T$ be a subset of $S$. The closure of $T$ relative to the closure system $\mathcal{K}$ is the set $K(T) = \bigcap \{U \in \mathcal{K} \mid T \subseteq U\}$.

For every set $T$ the collection $\mathcal{C}_T = \{U \in \mathcal{K} \mid T \subseteq U\}$ is non-empty because it includes at least $S$. The set $\bigcap \mathcal{C}_T$ is denoted by $K(T)$ and is referred to as the closure of $T$.

To emphasize that the closure of $T$ is computed relative to the closure system $\mathcal{K}$ we may denote this closure by $K_{\mathcal{K}}(T)$. 
Example

A subset \( E \) of \( \mathbb{R} \) is said to be symmetric if \( x \in E \) if and only if \( -x \in E \). Let \( \{ E_i \mid i \in I \} \) be a collection of symmetric subsets of \( \mathbb{R} \). It is easy to see that \( \bigcap \{ E_i \mid i \in I \} \) is a symmetric set. Note that \( \mathbb{R} \) itself is symmetric. Thus, the collection \( \mathcal{E} \) of symmetric subsets of \( \mathbb{R} \) is a closure system. For a subset \( T \) of \( \mathbb{R} \) the set \( K_{\mathcal{E}}(T) \) is the smallest symmetric set that includes \( T \).
Let $A$ and $B$ be two finite sets. It is easy to verify that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$ 

In this section we discuss a generalization of this Equality known as the **inclusion-exclusion principle**.

**Definition**

Let $S$ be a set and let $U$ be a subset of $S$. The *indicator* of $U$ is the function $I_U : S \rightarrow \{0, 1\}$ given by

$$I_U(x) = \begin{cases} 
1 & \text{if } x \in U, \\
0 & \text{if } x \notin U.
\end{cases}$$
If $U$ and $V$ are two subsets of a finite set $S$ such that $V \subseteq U$, then the function $I$ defined by $I(x) = I_U(x) - I_V(x)$ for $x \in S$ is an indicator function, namely the indicator function of the subset $U - V$ of $S$. Let $a$ and $b$ be two numbers that belong to the set $\{-1, 1\}$ such that the function $I_{ab}$ defined by

$$I_{ab}(x) = aI_U(x) + bI_V(x)$$

for $x \in S$ is the indicator function of a subset $W$ of the set $S$. We need to examine conditions under which $I_{ab}$ is the indicator function of a set.
Since $I_{ab}(x) \in \{0, 1\}$ and $I_{ab}(x) = aI_U(x) + bI_V(x)$, the following cases are possible:

- If $a = b = 1$, then we have $U \cap V = \emptyset$; otherwise (that is, if $x \in U \cap V$) we would have $aI_U(x) + bI_V(x) = 2$ and this would prevent $I_{ab}$ from being an indicator function. Clearly, in this case, $W = U \cup V$.
- If $a = 1$ and $b = -1$, we must have $I_V(x) \leq I_U(x)$ for every $x \in S$, which implies $V \subseteq U$. Thus, $W = U - V$.
- The case where $a = -1$ and $b = 1$ is similar to the previous case, and we have $W = V - U$.
- The case when $a = -1$ and $b = -1$ is possible only if $U = V = \emptyset$. In this case, $W = \emptyset$.

Note that in all these cases we have $|W| = a|U| + b|V|$. This observation is generalized by the next statement.
Theorem

Let $U_0, \ldots, U_{n-1}$ be $n$ subsets of a finite set $S$, where $n \geq 2$, and let $(a_0, \ldots, a_{n-1}) \in \text{Seq}_n(\{-1, 1\})$ be a sequence of $n$ numbers such that the function $I : S \to \{0, 1\}$ defined by

$$I(x) = a_0 I_{U_0}(x) + \cdots + a_{n-1} I_{U_{n-1}}(x)$$

for $x \in S$ is the indicator function of a subset $W$ of $S$. Then,

$$|W| = a_0 |U_0| + \cdots + a_{n-1} |U_{n-1}|.$$
Proof

If \( W \) is a subset of \( S \), then \( \sum_{x \in S} I_W(x) = |W| \) because for each \( x \in S \) its contribution to the sum \( \sum_{x \in S} I_W(x) \) is equal to 1 if and only if \( x \in W \). Therefore, if \( I_W(x) = \sum_{i=0}^{n-1} a_i I_{U_i}(x) \) for \( x \in S \), we have

\[
|W| = \sum_{x \in S} I_W(x) = \sum_{x \in S} \sum_{i=0}^{n-1} a_i I_{U_i}(x) = \sum_{i=0}^{n-1} \sum_{x \in S} a_i I_{U_i}(x)
\]

\[
= \sum_{i=0}^{n-1} a_i \sum_{x \in S} I_{U_i}(x) = \sum_{i=0}^{n-1} a_i |U_i|.
\]
Corollary

(Principle of Inclusion-Exclusion) Let \( A_0, \ldots, A_{n-1} \) be \( n \) finite sets, where \( n \geq 2 \). We have

\[
\left| \bigcup_{i=0}^{n-1} A_i \right| = \sum_{0 \leq i \leq n-1} |A_i| - \sum_{0 \leq i_1 < i_2 \leq n-1} |A_{i_1} \cap A_{i_2}| + \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots + (-1)^{n+1} |A_0 \cap \cdots \cap A_{n-1}|.
\]
Proof

Suppose that \( A_i \subseteq S \) for \( 0 \leq i \leq n - 1 \), where \( S \) is a finite set. For \( x \in S \), we have \( x \notin A = \bigcup_{i=0}^{n-1} A_i \) if and only if \( x \notin A_i \) for \( 0 \leq i \leq n - 1 \). This is equivalent to writing

\[
1 - l_A(x) = (1 - l_{A_{i_0}}(x)) \cdots (1 - l_{A_{i_{n-1}}}(x))
\]

for every \( x \in S \). This equality is, in turn, equivalent to

\[
l_A(x) = \sum_{i=0}^{n-1} l_{A_i}(x) - \sum_{0 \leq i_1 < i_2 \leq n-1} l_{A_{i_1}}(x)l_{A_{i_2}}(x)
\]

\[
+ \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} l_{A_{i_1}}(x)l_{A_{i_2}}(x)l_{A_{i_3}}(x) - \cdots + (-1)^{n+1} l_{A_0}(x) \cdots l_{A_{n-1}}(x)
\]

\[
= \sum_{i=0}^{n-1} l_{A_i}(x) - \sum_{0 \leq i_1 < i_2 \leq n-1} l_{A_{i_1} \cap A_{i_2}}(x)
\]

\[
+ \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} l_{A_{i_1} \cap A_{i_2} \cap A_{i_3}}(x) - \cdots + (-1)^{n+1} l_{A_0 \cap \cdots \cap A_{n-1}}(x).
\]
Corollary

Let $A_0, \ldots, A_{n-1}$ be $n$ finite sets, where $n \geq 2$, and let $S = \bigcup_{i=0}^{n-1} A_i$. We have

$$\left| \bigcap_{i=0}^{n-1} A_i \right| = |S| - \sum_{0 \leq i \leq n-1} |A_i| + \sum_{0 \leq i_1 < i_2 \leq n-1} |A_{i_1} \cap A_{i_2}|$$
$$- \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| + \cdots + (-1)^n |A_0 \cap \cdots \cap A_{n-1}|$$

Proof: This follows immediately from the previous Corollary by observing that

$$\left| \bigcap_{i=0}^{n-1} A_i \right| = |S| - \left| \bigcup_{i=0}^{n-1} A_i \right|.$$
Definition

A relation on the set $S$ is a set of ordered pairs of $S$.

The set of relations on $S$ is denoted by $\text{rel}(S)$. Since relations on $S$ are sets of pairs on $S$ they can be involved in the usual set-theoretical operations: union, intersection, difference, etc. If $\rho, \sigma \in \text{rel}(S)$, the union, intersection, and difference of $\rho$ and $\sigma$ are denoted by $\rho \cup \sigma$, $\rho \cap \sigma$, and $\rho - \sigma$, respectively. Also, $\rho \subseteq \sigma$ denotes the inclusion of the set of pairs $\rho$ into the set of pairs $\sigma$. 
Two important relations on $S$ are the *diagonal relation* 

$$\iota_S = \{(x, x) \mid x \in S\},$$

and the *total relation*

$$\theta_S = \{(x, y) \mid x, y \in S\}.$$

**Definition**

Let $\rho, \sigma \in \text{rel}(S)$. The *product* of $\rho$ and $\sigma$ is the relation $\rho\sigma$ given by

$$\rho\sigma = \{(x, z) \in \text{Seq}_2(S) \mid (x, y) \in \rho \text{ and } (y, z) \in \sigma\}.$$
Definition

A relation $\rho \in \text{rel}(S)$ is:

- **reflexive**, if $\iota_S \subseteq \rho$;
- **symmetric**, if $(x, y) \in \rho$ is equivalent to $(y, x) \in \rho$;
- **antisymmetric**, if $(x, y) \in \rho$ and $(y, x) \in \rho$ implies $x = y$;
- **transitive**, if $(x, y), (y, z) \in \rho$ implies $(x, z) \in \rho$,

for all $x, y, z \in S$.

If $\rho \in \text{rel}(S)$, the **inverse** of $\rho$ is the relation

$$\rho^{-1} = \{(y, x) \in S \times S \mid (x, y) \in \rho\}.$$
The $n^{th}$ power of a relation $\rho$, where $\rho \subseteq S \times S$ is defined inductively as

$$\rho^0 = \iota_S,$$
$$\rho^{n+1} = \rho^n \rho$$

for $n \geq 0$.

If $\rho$ is a relation on $S$, then $(x, x) \in \rho^0$ for every $x \in S$. An easy argument by induction on $n \in \mathbb{N}$ shows that $(x, y) \in \rho^n$ if and only if there exists a sequence $z = (z_0, z_1, \ldots, z_n)$ of length $n + 1$ such that $x = z_0$, $(z_i, z_{i+1}) \in \rho$ for $0 \leq i \leq n - 1$ and $z_n = y$. 
Properties of relations can be expressed using the operations just introduced. For example, a relation \( \rho \) on a set \( S \) is symmetric if and only if \( \rho^{-1} = \rho \); a relation \( \rho \) is transitive if \( \rho^2 \subseteq \rho \).

**Definition**

An *equivalence relation* on a set \( S \) is a relation \( \rho, \rho \subseteq S \times S \) that is reflexive, symmetric, and transitive. The set of equivalence relations on \( S \) is denoted by \( \text{EQ}(S) \).
Example

Both $\nu_S$ and $\theta_S$ are equivalence relations on $S$; moreover, for any equivalence $\rho \in \text{EQ}(S)$ we have $\nu_S \subseteq \rho \subset \theta_S$. 
Example

Let $m$ be a positive integer. Define the relation $\equiv_m$ on $\mathbb{Z}$ as consisting of those pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ if $p - q = km$ for some $k \in \mathbb{Z}$. In other words, we have $(p, q) \in \equiv_m$ if $p - q$ is divisible by $m$.

Note that $(r, r) \in \equiv_m$ because $r - r = 0$ is divisible by $m$. If $p - q = km$ for some $k \in \mathbb{Z}$, then $q - p = (-k)m$, so $(p, q) \in \equiv_m$ implies $(q, p) \in \equiv_m$. Finally suppose that $(p, q) \in \equiv_m$ and $(q, s) \in \equiv_m$. Since $p - q = km$ and $q - s = hm$, we have $p - s = (k + h)m$, hence $(p, s) \in \equiv_m$. Thus, $\equiv_m$ is an equivalence relation on $\mathbb{Z}$. 
Following common practice, for an equivalence \( \rho \) on a set \( S \) and for \( x, y \in S \) we write \( x \rho y \) for \( (x, y) \in \rho \).

**Definition**

Let \( \rho \) be an equivalence relation on a set \( S \). The *equivalence class* of an element \( x \) of \( S \) is the set

\[
[x]_\rho = \{ u \in S \mid (x, u) \in \rho \}.
\]

By the reflexivity of \( \rho \), \( (x, x) \in \rho \) for every \( x \in S \). Thus, \( x \in [x]_\rho \), hence each equivalence class is non-empty.
Lemma

Let $\rho$ be an equivalence relation on a set $S$. We have $y \in [x]_{\rho}$ if and only if $[y]_{\rho} = [x]_{\rho}$.
Proof

Suppose that $y \in [x]_\rho$ and that $u \in [y]_\rho$. Then, we have $(x, y) \in \rho$ and $(y, u) \in \rho$. By transitivity, $(x, u) \in \rho$, that is, $u \in [x]_\rho$, which implies $[y]_\rho \subseteq [x]_\rho$.

If $v \in [x]_\rho$, then $(x, v) \in \rho$. Since $(x, y) \in \rho$, by the symmetry and transitivity of $\rho$ we obtain $(y, v) \in \rho$, hence $v \in [y]_\rho$, so $[x]_\rho \subseteq [y]_\rho$. This implies $[x]_\rho = [y]_\rho$.

Conversely, if $[y]_\rho = [x]_\rho$, we have $y \in [x]_\rho$ because $y \in [y]_\rho$. 
Theorem

Let $\rho$ be an equivalence relation on a set $S$. If $[x]_\rho \neq [y]_\rho$, then $[x]_\rho \cap [y]_\rho = \emptyset$.

Proof.

Let $x, y \in S$ be such that $[x]_\rho \neq [y]_\rho$ and suppose that $z \in [x]_\rho \cap [y]_\rho$. Since $z \in [x]_\rho$ we have $[z]_\rho = [x]_\rho$; similarly, since $z \in [y]_\rho$ we have $[z]_\rho = [y]_\rho$, which means that $[x]_\rho = [y]_\rho$. This contradicts the hypothesis.
Definition

Let $S$ be a non-empty set. A partition on $S$ is a non-empty collection $\pi = \{B_i \mid i \in I\}$ such that

- $B_i \neq \emptyset$ for $i \in I$;
- $i, j \in I$ and $i \neq j$ implies $B_i \cap B_j = \emptyset$;
- $\bigcup_{i \in I} B_i = S$.

The sets $B_i$ are the blocks of the partition $\pi$.

The set of partitions of a set $S$ is denoted by $\text{PART}(S)$; the set of partitions of $S$ that have $k$ blocks, where $1 \leq k \leq |S|$ is denoted by $\text{PART}_k(S)$.

The partitions in $\text{PART}_2(S)$ are referred to as bipartitions.

Clearly, $\text{PART}(S) = \bigcup_{k=1}^{|S|} \text{PART}_k(S)$. 

Example

The partition of a set $S$ that consists of all singletons \{\{x\}\}, where $x \in S$ is denoted by $\alpha_S$; the partition of $S$ that contains one block, namely $S$, is denoted by $\omega_S$. We have $\text{PART}_{|S|}(S) = \{\alpha_S\}$ and $\text{PART}_1(S) = \{\omega_S\}$. 
Example

Let \( \rho \) be an equivalence relation on a set \( S \). The set of equivalence classes of \( \rho \) is a partition of the set \( S \). Indeed, we saw that \( S = \bigcup_{x \in S} [x]_{\rho} \), no equivalence class is empty and, as we saw, any two equivalence classes are disjoint.

The set of equivalence classes of an equivalence relation is known as the *quotient set* of \( S \) by \( \rho \) and is denoted by \( S/\rho \). The partition generated by the equivalence relation is also denoted by \( \pi_{\rho} \).
Example

Let \( m \in \mathbb{P} \) and let \( B_i \) be the set of all members \( n \) of \( \mathbb{P} \) such that the remainder of the division of \( n \) by \( m \) equals \( i \), where \( 0 \leq i \leq m - 1 \). It is immediate that the collection \( \{B_0, B_1, \ldots, B_{m-1}\} \) is a partition of the set \( \mathbb{P} \). For instance, if \( m = 3 \), we have \( B_0 = \{3, 6, 9, 12 \ldots\} \), \( B_1 = \{1, 4, 7, 10, \ldots\} \), and \( B_2 = \{2, 5, 8, 11, \ldots\} \).
Theorem

Let $\pi = \{B_i \mid i \in I\}$ be a partition of the set $S$. The relation $\rho_\pi$ defined by

$$\rho_\pi = \{(x, y) \in S \times S \mid \{x, y\} \subseteq B_i \in \pi\}$$

is an equivalence on $S$. 
Proof

Each $x$ belongs to a block $B_i$ of $\pi$, so $(x, x) \in \rho_\pi$ for every $x \in S$, which means that $\rho_\pi$ is reflexive.

If $(x, y) \in \rho_\pi$, then $\{x, y\} \subseteq B_i$, which obviously implies $(y, x) \in \rho_\pi$, so $\rho_\pi$ is symmetric.

Finally, if $(x, y) \in \rho_\pi$ and $(y, z) \in \rho_\pi$, there exist $B_i, B_j \in \pi$ such that $\{x, y\} \subseteq B_i$ and $\{y, z\} \subseteq B_j$. Thus, $B_i \cap B_j \neq \emptyset$ (because both contain $y$), which implies $B_i = B_j$. Therefore, $\{x, z\} \subseteq B_i = B_j$, hence $(x, z) \in \rho_\pi$, which allows us to conclude that $\rho_\pi$ is an equivalence relation.
Corollary

Let $\pi \in \text{PART}(S)$ and let $\rho \in \text{EQ}(S)$ $\rho = \rho_{\pi \rho}$ and $\pi = \pi_{\rho \pi}$.

Proof.

The equalities follow easily from the definitions of $\pi_{\rho}$ and $\rho_{\pi}$.
Example

Note that $\pi_{\iota_S} = \alpha_S$, $\pi_{\theta_S} = \omega_S$ and $\rho_{\alpha_S} = \iota_S$, $\rho_{\omega_S} = \theta_S$.

We write $x \equiv y(\pi)$ to denote that $(x, y) \in \rho_{\pi}$. 
Denote by \((x)_n\) the \(n\)-degree polynomial

\[(x)_n = x(x - 1) \cdots (x - n + 1).\]

The coefficients of this polynomial

\[(x)_n = s(n, n)x^n + s(n, n - 1)x^{n-1} + \cdots + s(n, i)x^i + \cdots + s(n, 0)\]

are known as the Stirling numbers of the first kind.
Theorem

We have:

\[ s(n, 0) = 0, \]
\[ s(n, n) = 1, \]
\[ s(n + 1, k) = s(n, k - 1) - ns(n, k). \]

Proof.

The verification of the first two equalities is immediate. The third equality follows by observing that \((x)^{n+1} = (x)^n(x - n)\) and seeking the coefficient of \(x^k\) on both sides.
Let $S$ be a set having $n$ elements. We are interested in the number of partitions of $S$ that have $k$ blocks. We begin by counting the number of onto functions of the form $f : A \rightarrow B$, where $|A| = n$, $|B| = k$, and $n \geq k$.

**Lemma**

Let $A$ and $B$ be two sets, where $|A| = n$, $|B| = k$, and $n \geq k$. The number of surjective functions from $A$ to $B$ is given by

$$
\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k - j)^n .
$$
Proof

There are $k^n$ functions of the form $f : A \rightarrow B$.

We begin by determining the number of functions that are not surjective. Suppose that $B = \{b_1, \ldots, b_k\}$, and let $F_j = \{f : A \rightarrow B \mid b_j \not\in f(A)\}$ for $1 \leq j \leq k$. A function is not surjective if it belongs to one of the sets $F_j$. Thus, we need to evaluate $|\bigcup_{j=1}^k F_j|$. By using the inclusion-exclusion principle, we can write:

\[
|\bigcup_{j=1}^k F_j| = \sum_{j_1=1}^k |F_{j_1}| - \sum_{j_1, j_2=1}^k |F_{j_1} \cap F_{j_2}| + \sum_{j_1, j_2, j_3=1}^k |F_{j_1} \cap F_{j_2} \cap F_{j_3}| - \cdots + (-1)^k |F_1 \cap F_2 \cap \cdots \cap F_k|.
\]
Proof (cont’d)

Note that the set \( |F_{j-1} \cap F_j \cap \cdots \cap F_p| \) is actually the set of functions defined on \( A \) with values in the set \( B - \{ y_{j_1}, y_{j_2}, \ldots, y_{j_p} \} \), and there are \((k - p)^n\) such functions. Since there are \( \binom{k}{p} \) choices for the set \( \{j_1, j_2, \ldots, j_p\} \), it follows that there are

\[
\binom{k}{1}(k - 1)^n - \binom{k}{2}(k - 2)^n + \binom{k}{3}(k - 3)^n - \cdots + (-1)^k \binom{k}{k - 1}
\]

functions that are not surjective.
Proof (cont’d)

Thus, we can conclude that there are

\[\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n\]

\[= k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \cdots + (-1)^{k-1}\binom{k}{k-1}\]

surjective functions from \(A\) to \(B\).
The number of partitions of a set $S$ that have $k$ blocks ($k \leq n$) is given by

$$\frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n.$$
Proof

Note that there are $k!$ distinct onto functions that have the same kernel partition. Indeed, given a surjective function $f : A \rightarrow B$, one can obtain a function $g$ that has the same partition as $f$ by defining $g(a) = p(f(a))$, where $p$ is a permutation of the set $B$, that is, a bijection $p : B \rightarrow B$. Since there are $k!$ such bijections, it follows that the number of partitions is $\frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k - j)^n$. 
The numbers $S(n, k)$ defined by

$$S(n, k) = \begin{cases} \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n & \text{if } n \geq k > 0, \\ 1 & \text{if } n = k = 0, \\ 0 & \text{in other cases}. \end{cases}$$

for $n, k \in \mathbb{N}$ and are known as the *Stirling numbers of the second kind.*
Example

Note that $S(n, 1) = 1$ and $S(n, n) = 1$ because only one partition of a set with $n$ elements, $\omega_S$, has one block, and only one partition of a set with $n$ elements, $\alpha_S$ has $n$ blocks which are singletons.

The number of partitions of a 4-element set having two blocks is

$$S(4, 2) = \frac{1}{2!} \sum_{j=0}^{1} \binom{2}{j} (2 - j)^4$$

$$= \frac{1}{2!} \left( \left( \binom{2}{0} \cdot 2^4 - \binom{2}{1} \cdot 1^4 \right) \right) = 7.$$ 

Namely, these partitions are:

$$\{\{1\}, \{2, 3, 4\}\}, \{\{2\}, \{1, 3, 4\}\}, \{\{3\}, \{1, 2, 4\}\}, \{\{4\}, \{1, 2, 3\}\}, \{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}.$$
We claim that

\[ S(n, k) = kS(n - 1, k) + S(n - 1, k - 1). \]

Indeed, observe that a partition \( \pi \) of the set \( \{1, \ldots, n - 1\} \) can be transformed into a partition of \( \{1, \ldots, n\} \) by adjoining \( n \) to one of the blocks of \( \pi \) or by increasing the number of blocks by 1 and making \( \{n\} \) a block.
Theorem

For every \( n \geq 1 \) we have \( m^n = \sum_{j=1}^{n} S(n, j)(m)_j \).
Proof

Let $A$ and $B$ be two finite sets such that $|A| = n$ and $|B| = m$. There are $m^n$ functions $f : A \longrightarrow B$. These functions can be classified depending on the size of their range $f(A)$. If $g : A \longrightarrow B$ is a function such that $|g(A)| = j$, then $g$ can be regarded as a surjection from $A$ to $g(A)$. Since there are $j!S(n,j)$ such surjective functions and there are $\binom{m}{j}$ subsets of $B$ that have $j$ elements, we can write

$$m^n = \sum_{j=1}^{m} \binom{m}{j} j!S(n,j)$$

$$= \sum_{j=1}^{m} m(m-1) \cdots (m-j+1)S(n,j) = \sum_{j=1}^{m} (m)_n S(n,j)$$

for every $m \geq 1$. 
The **Bell number** $B_n$ is the total number of partitions of a set of $n$ objects, that is,

$$B_n = \sum_{k=1}^{n} S(n, k).$$

**Example**

For $n = 4$, we have shown that there exist 7 partitions having two blocks, one partition with one block and one partition with 4 blocks. It is easy to see that there are 6 partitions with 3 blocks, so $B_4 = 1 + 7 + 6 + 1 = 15$.

The first 10 values of the Bell numbers are given below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
<td>877</td>
<td>4140</td>
<td>21147</td>
<td>115975</td>
</tr>
</tbody>
</table>
Definition

A relation $\rho$ is a **partial order** on a set $S$ if $\rho$ is reflexive, antisymmetric and transitive.

A **partially ordered set** (or, a **poset**) is a pair $(S, \rho)$, where $\rho$ is a partial order on $S$.

In general, we denote partial orders using the symbol “$\leq$” or similar symbols; furthermore, instead of writing $(x, y) \in \leq$, we write $x \leq y$. 
Example

Let $T$ be a set. The set of subsets of $T$, $\mathcal{P}(T)$ equipped with the set inclusion “$\subseteq$” yields the poset $(\mathcal{P}(T), \subseteq)$. 
Example

The pair \((\mathbb{P}, |)\), where “|” is the divisibility relation is a poset defined by \(p|q\) if there exists \(k \in \mathbb{P}\) such that \(q = pk\). Indeed, we have \(p|p\) for every \(p \in \mathbb{P}\), so “|” is reflexive. If \(p|q\) and \(q|p\), we have \(q = pk\) and \(p = qh\), hence \(hk = 1\) which implies \(h = k = 1\). Thus, \(p = q\), which shows that “|” is antisymmetric. Finally, if \(p|q\) and \(q|r\) we have \(q = pk\) and \(r = qh\) for some \(k, h \in \mathbb{P}\). Thus, \(r = pkh\), so \(p|r\).
Example

Let \((\mathbb{P}, |)\) be the poset of positive numbers equipped with the divisibility relation. We have \(p \prec y\) if \(x\) is none of the largest divisors of \(y\). For example, we have \(6 \prec 12\) because there is no number \(z\) distinct from 6 and 12 such that \(6|z\) and \(z|12\). Note that \(3|12\) but \(3 \not\prec 12\).
Finite posets can be represented graphically using *Hasse diagrams*.

- Each element is represented by a dot.
- If $x, y$ are elements of a poset $(P, \leq)$ and $x \prec y$, then the dot representing $y$ is placed at a greater height than $x$ and a link between the dots is drawn.
The Hasse diagram of the poset \((\mathcal{P}(\{1, 2, 3\}), \subseteq)\) equipped with the inclusion relation is given below:
Example

The Hasse diagram of the poset \((\{1, \ldots, 12\}, \mid)\) equipped with the divisibility relation is given below:
If \((S, \rho)\) is a poset and \(T \subseteq S\), it is easy to see that the relation 
\[ \rho_T = \rho \cap (T \times T) \]
is itself a partial order; we will refer to it as the *trace of \((S, \rho)\) on \(T\).* Often, we will use the same symbol \(\rho\) instead of \(\rho_T\) to 
denote the partial order on \(T\).

**Example**

Let \(S \subseteq \mathcal{P}\) be the set \(\{1, 2, 3, 4, 5, 6\}\). The trace of \(\mathcal{P}\) on \(S\) consists of the pairs:

\[(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6).\]
Definition

A *totally ordered set* is a pair \((S, \rho)\), where \(\rho\) is a partial order with the additional property that for all \(x, y \in S\) we have either \((x, y) \in \rho\), or \((y, x) \in \rho\). The relation \(\rho\) is referred to as a *total order*.

Example

The real numbers \(\mathbb{R}\) equipped with the standard less-than-or-equal relation \(\leq\) is a totally ordered set.
Definition

A sequence $x \in \text{Seq}(S)$ is a subsequence of a sequence $y \in \text{Seq}(S)$, if $y = uxv$ for some sequences $u, v \in \text{Seq}(S)$. This is denoted by $x \sqsubseteq y$.

Example

Let $S = \{0, 1\}$. The sequence $y = 1011$ is a subsequence of $x = 010110110101100$.

The relation “$\sqsubseteq$” is a partial order on $\text{Seq}(S)$. 
Definition

Let \((P, \leq)\) be a poset. An element \(y\) covers an element \(x\) of \(P\) if \(x \leq y\) and there is no \(z \in P\), \(z \neq x\) and \(z \neq y\) such that \(x \leq z \leq y\). We denote the fact that \(y\) covers \(x\) by \(x \prec y\).
Definition

Let \((S, \leq)\) be a poset and let \(T\) be a subset of \(S\). The *set of upper bounds* of \(T\) is the set

\[
T^s = \{y \in S \mid \text{for all } x \in T \text{ we have } x \leq y\}.
\]

The *set of lower bounds* of \(T\) is the set

\[
T^i = \{y \in S \mid \text{for all } x \in T \text{ we have } y \leq x\}.
\]
If $T_1, T_2$ are two subsets of $S$, $T_1 \subseteq T_2$ implies $T_2^s \subseteq T_1^s$ and $T_2^i \subseteq T_1^i$.

**Theorem**

Let $(S, \leq)$ be a poset and let $T$ be a subset of $S$. The sets $T \cap T^s$ and $T \cap T^i$ contain at most one element of $S$.

**Proof.**

Suppose that $x, y \in T \cap T^s$. Since $x \in T$ and $y \in T^s$, it follows that $x \leq y$. On other hand, since $x \in T^s$ and $y \in T$ we have $y \leq x$. Therefore, $x = y$, which implies that the set $T \cap T^s$ contains at most one element. The argument for $T \cap T^i$ is similar. □
**Definition**

Let \((S, \leq)\) be a poset and let \(T\) be a subset of \(S\). If \(T \cap T^s = \{u\}\), then \(u\) is the *largest element* of set \(T\).

If \(T \cap T^i = \{v\}\), then \(v\) is the *least element* of set \(T\).

**Example**

Not every subset of a poset has a least or a greatest element. The subset \(\{1, 2, 3, 6\}\) of the poset \((\{1, \ldots, 12\}, \mid)\) considered before has 1 as its least element and 6 as the largest element. In contrast, the set \(\{4, 5, 6\}\) has neither a least nor a largest element.
If \( T \) is a subset of a poset \((S, \leq)\) we will consider the sets \((T^s)^i\) and \((T^i)^s\) denoted by \(T^si\) and \(T^is\), respectively.

Observe that the set \(T^s \cap T^si = T^s \cap (T^s)^i\) may contain at most one element, by a previous observation applied to the set \(T^s\). Similarly, the set \(T^i \cap T^is\) may contain at most one element.
Definition

Let \((S, \leq)\) be a poset and let \(T\) be a subset of \(S\). If \(T^s \cap T^{si} = \{u\}\), \(u\) is the supremum of the set \(T\).

If \(T^i \cap T^{is} = \{v\}\), \(v\) is the infimum of \(T\).

The supremum and infimum of a set \(T\) (if they exist) are unique and are denoted by sup \(T\) and inf \(T\), respectively.
Example

In the poset \((\mathcal{P}(T), \subseteq)\) introduced in before, for every \(C \in \mathcal{P}(X)\) we have

\[
\inf C = \bigcap C \quad \text{and} \quad \sup C = \bigcup C.
\]
Example

In the poset \((\mathbb{P}, |)\), we have

\[
\inf\{p, q\} = \gcd(p, q) \quad \text{and} \quad \sup\{p, q\} = \text{lcm}(p, q),
\]

where \(\gcd(p, q)\) is the greatest common divisor of \(p\) and \(q\), and \(\text{lcm}(p, q)\) is the least common multiple of \(p\) and \(q\).
Definition

A poset \((S, \leq)\) is a \textit{lattice} if for every two elements \(x, y \in S\) there exist \(\inf\{x, y\}\) and \(\sup\{x, y\}\). If \((S, \leq)\) is a lattice we use the notations

\[x \wedge y = \inf\{x, y\}\] and \[x \vee y = \sup\{x, y\}\].

The element \(x \wedge y\) is referred to as the \textit{meet} of \(x\) and \(y\); \(x \vee y\) is the \textit{join} of \(x\) and \(y\).

A poset \((S, \leq)\) is a \textit{complete lattice} if for every \(X \in \mathcal{P}(S)\) there exist \(\inf X\) and \(\sup X\).
Example

- $(\mathcal{P}, |)$ is a lattice;
- $(\mathcal{P}(T), \subseteq)$ is a complete lattice.
Note that if \((S, \subseteq)\) is a complete lattice and \(S \neq \emptyset\), then this poset has a least element \(0_S = \inf S\), and a greatest element \(1_S = \sup S\).

**Theorem**

Let \((S, \subseteq)\) be a complete lattice and let \(W\) be a subset of \(S\) such that \(1_S \in W\) and \(T \subseteq W\) implies that \(\inf T\) in \(S\) belongs to \(W\). Then \(W\) is a complete lattice.
Proof

For every nonvoid subset $T$ of $W$, $\inf T \in W$ and is the infimum of $T$ in $S$. Let $U$ be a subset of $W$ defined as $U = T^s$. We have $U \neq \emptyset$ because $1_S \in W$. Then, $\inf U \in W$ is also an upper bound of $T$, and is actually the least upper bound of $U$. Thus, $(W, \leq)$ is a complete lattice.
Corollary

Let $\mathcal{K}$ be a closure system on a set $S$. The subsets of $S$ in $\mathcal{K}$ form a complete lattice in which $\inf C = \bigcap C$ and $\sup C = \bigcap \{ T \in \mathcal{P} \mid C \subseteq T \text{ for every } C \in \mathcal{K} \}$. 
Let $\pi, \sigma$ be two partitions of $S$. We write $\pi \leq \sigma$ if each block $B$ of $\pi$ is included in a block $C$ of $\sigma$.

**Theorem**

The pair $(\text{PART}(S), \leq)$ is a partially ordered set.
Proof

The relation “⩽” is obviously reflexive. Suppose that we have both $\pi \leq \sigma$ and $\sigma \leq \pi$. Then, a block $B$ of $\pi$ is included in a block $C$ of $\sigma$, and $C$, in turn, is included in a block $B'$ of $C$. Thus, $B \subseteq C \subseteq B'$, which implies $B = C = B'$ because no block of $\pi$ can be included into another block. Thus, $\pi \subseteq \sigma$. In the same manner, starting from a block $C$ of $\sigma$ we can show that $\sigma \subseteq \pi$, so $\pi = \sigma$. This shows that the relation “⩽” is antisymmetric. It is immediate that “⩽” is transitive.
Let $\pi, \sigma \in \text{PART}(S)$ be two partitions, $\pi = \{B_i \mid i \in I\}$ and $\sigma = \{C_j \mid j \in J\}$. We have $\pi \leq \sigma$ if and only if for each $j \in J$ there exists a subset $I_j$ of $I$ such that $C_j = \bigcup\{B_i \mid i \in I_j\}$.

Suppose that $\pi \leq \sigma$ and let $C \in \sigma$. Suppose that $B \cap C \neq \emptyset$. Since each block $B$ of $\pi$ is included in a block $C'$ of $\sigma$ we must have $C' = C$ because, otherwise $C'$ and $C$ would have a non-empty intersection. Thus, if a block $B$ of $\pi$ has a non-empty intersection with a block $C$ of $\sigma$ we must have $B \subseteq C$. This implies that a block $C$ of $\sigma$ is a union of block of $\pi$. The converse implication is immediate.
Example

If $\pi \in \text{PART}(S)$ we have $\alpha_S \leq \pi \leq \omega_S$. Thus, $\alpha_S$ is the smallest element of $(\text{PART}(S), \leq)$ and $\omega_S$ is its largest element.
Definition

Let $\pi, \sigma$ be two partitions of a set $S$. The partition $\sigma$ covers $\pi$ if $\pi < \sigma$ and there is no partition $\tau \in \text{PART}(S)$ such that $\pi < \tau < \sigma$. 
Theorem

Let $\pi, \sigma$ be two partitions of a set $S$. The partition $\sigma$ covers $\pi$ if and only if there exists a block $C$ of $\sigma$ that is the union of two blocks $B$ and $B'$ of $\pi$, and every other block of $\sigma$ that is distinct of $C$ is a block of $\pi$. 
Proof

Suppose that $\sigma$ is a partition that covers the partition $\pi$. Since $\pi \leq \sigma$, every block of $\sigma$ is a union of blocks of $\pi$. Suppose that there exists a block $E$ of $\sigma$ that is the union of more than two blocks of $\pi$; that is, $E = \bigcup\{B_i \mid i \in I\}$, where $|I| \geq 3$, and let $B_{i_1}, B_{i_2}, B_{i_3}$ be three blocks of $\pi$ included in $E$. Consider the partitions

$$\sigma_1 = \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1}, B_{i_2}, B_{i_3}\},$$
$$\sigma_2 = \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1} \cup B_{i_2}, B_{i_3}\}.$$

It is easy to see that $\pi \leq \sigma_1 < \sigma_2 < \sigma$, which contradicts the fact that $\sigma$ covers $\pi$. Thus, each block of $\sigma$ is the union of at most two blocks of $\pi$. 
Proof (cont’d)

Suppose that $\sigma$ contains two blocks $C'$ and $C''$ that are unions of two blocks of $\pi$, namely $C' = B_{i_0} \cup B_{i_1}$ and $C'' = B_{i_2} \cup B_{i_3}$. Define the partitions

$$\sigma' = \{ C \in \sigma \mid C \not\in \{C', C''\} \} \cup \{ C', B_{i_2}, B_{i_3} \},$$

$$\sigma'' = \{ C \in \sigma \mid C \not\in \{C', C''\} \} \cup \{ B_{i_1}, B_{i_2}, C'' \}.$$

Since $\pi < \sigma'$, $\sigma'' < \sigma$, this contradicts the fact that $\sigma$ covers $\pi$. Thus, we obtain the conclusion of the theorem.
Example

The Hasse diagram of \((\text{PART} (\{1, 2, 3\}), \subseteq)\) is given below:

\[
\begin{align*}
\{\{1, 2, 3\}\} \\
\{\{1, 2\}, \{3\}\} & \quad \{\{1\}, \{2, 3\}\} & \quad \{\{1, 3\}, \{2\}\} \\
\{\{1\}, \{2\}, \{3\}\}
\end{align*}
\]
Theorem

The posets \((EQS(S), \subseteq)\) and \((PART(S), \subseteq)\) are isomorphic.

Let \(f : EQS(S) \rightarrow PART(S)\) be the mapping defined by \(f(\rho) = S/\rho\). We need to show that \(f\) is a monotonic bijective mapping and that its inverse mapping \(f^{-1}\) is also monotonic.

The bijectivity of \(f\) follows immediately from the remarks that precede the theorem. Let \(\rho_0, \rho_1\) be two equivalences such that \(\rho_0 \subseteq \rho_1\) and let \(S/\rho_0 = \{B_i \mid i \in I\}, S/\rho_1 = \{C_j \mid j \in J\}\). Let \(B_i\) be a block in \(S/\rho_0\) and assume that \(B_i = [x]_{\rho_0}\). We have \(y \in B_i\) if and only if \((x, y) \in \rho_0\), so \((x, y) \in \rho_1\). Therefore, \(y \in [x]_{\rho_1}\), which shows that every block \(B \in S/\rho_0\) is included in a block \(C \in \rho_1\). This shows that \(f(\rho_0) \subseteq f(\rho_1)\), so \(f\) is indeed monotonic.
Let \( \{ \rho_i \mid i \in I \} \subseteq EQS(S) \) be a collection of equivalences. Then,
\[
\inf \{ \rho_i \mid i \in I \} = \bigcap_{i \in I} \rho_i.
\]

**Definition**

Let \( S \) be a set and let \( \rho, \tau \in EQS(S) \). A \((\rho, \tau)\)-alternating sequence that joins \( x \) to \( y \) is a sequence \((s_0, s_1, \ldots, s_n)\) such that \( x = s_0, y = s_n, (s_i, s_{i+1}) \in \rho \) for every even \( i \) and \((s_i, s_{i+1}) \in \tau \) for every odd \( i \), where \( 0 \leq i \leq n - 1 \).
Lemma

Let $S$ be a set and let $\rho, \tau \in EQS(S)$. If $s$ and $z$ are two $(\rho, \tau)$-alternating sequences joining $x$ to $y$ and $y$ to $z$, respectively, then there exists a $(\rho, \tau)$-alternating sequence that joins $x$ to $z$. 
Proof

Let \((s_0, \ldots, s_n)\) be a \((\rho, \tau)\)-alternating sequences joining \(x\) to \(y\) and \((w_0, \ldots, w_m)\) a \((\rho, \tau)\)-alternating sequences joining \(y\) to \(z\), where \(x = s_0, s_n = w_0 = y\) and \(w_m = z\). If \((s_{n-1}, s_n) \in \tau\), then the sequence \((s_0, \ldots, s_n, w_1, \ldots, w_m)\) is a \((\rho, \tau)\)-alternating sequence joining \(x\) to \(z\). Otherwise, that is, if \((s_{n-1}, s_n) \in \rho\), then taking into account the reflexivity of \(\tau\) we have \((s_n, w_0) = (s_n, s_n) \in \tau\). In this case, \((s_0, \ldots, s_n, s_n, w_1, \ldots, w_m)\) is a \((\rho, \tau)\)-alternating sequence joining \(x\) to \(z\).
Theorem

Let $S$ be a set and let $\rho, \tau \in EQS(S)$. If $\xi$ is the relation that consists of all pairs $(x, y) \in S \times S$ that can be joined by a $(\rho, \tau)$-alternating sequence, then $\xi = \sup\{\rho, \tau\}$. 
It is easy to verify that $\xi$ is indeed an equivalence relation. Note that we have both $\rho \subseteq \xi$ and $\tau \subseteq \xi$. Indeed, if $(x, y) \in \rho$, then $(x, y, y)$ is a $(\rho, \tau)$-alternating sequence joining $x$ to $y$. If $(x, y) \in \tau$, then $(x, x, y)$ is the needed alternating sequence.

Let $\zeta \in EQS(S)$ such that $\rho \subseteq \zeta$ and $\tau \subseteq \zeta$. If $(x, y) \in \xi$, and $(s_0, s_1, \ldots, s_n)$ is a $(\rho, \tau)$-alternating sequence such that $x = s_0$, $y = s_n$, then each pair $(s_i, s_{i+1})$ belongs to $\zeta$. By the transitivity property, $(x, y) \in \zeta$, so $\xi \subseteq \zeta$. This implies that $\xi = \sup\{\rho, \tau\}$.
If $\pi, \sigma \in \text{PART}(S)$ both the infimum and the supremum of the set $\{\pi, \sigma\}$ exist and their description follows from the corresponding results that refer to the equivalence relations. Namely, if $\pi, \sigma \in \text{PART}(S)$, where $\pi = \{B_i \mid i \in I\}$ and $\sigma = \{C_j \mid j \in J\}$, the partition $\inf\{\pi, \sigma\}$ exists and is given by

$$\inf\{\pi, \sigma\} = \{B_i \cap C_j \mid i \in I, j \in J \text{ and } B_i \cap C_j \neq \emptyset\}.$$  

The partition $\inf\{\pi, \sigma\}$ will be denoted by $\pi \wedge \sigma$. 
A block of the partition $\sup\{\pi, \sigma\}$, denoted by $\pi \vee \sigma$, is an equivalence class of the equivalence $\theta = \sup\{\rho_\pi \wedge \rho_\sigma\}$. We have $y \in [x]_\theta$ if there exists a sequence $(s_0, \ldots, s_n) \in \text{Seq}(S)$ such that $x = s_0$, $s_n = y$ and successive sets $\{s_i, s_{i+1}\}$ are included, alternatively, in a block of $\pi$ or in a block of $\sigma$. 