CS724: Topics in Algorithms
Set-Theoretical Preliminaries
Slide Set 1

Prof. Dan A. Simovici
Set Notations

For a set $S$ we denote by $\mathcal{P}(S)$ the set of its subsets. The set of all finite non-empty subsets of $S$ is denoted by $\mathcal{F}(S)$. For a finite set $S$ the number of elements of $S$ is denoted by $|S|$. The empty set is denoted by $\emptyset$.

We write $x \in S$ to denote the fact that $x$ is an element of the set $S$. The usual symbols are used to denote set-theoretical operations: $A \cup B$ is the union of the sets $A$ and $B$, $A \cap B$ is the intersection of the sets $A$ and $B$, and $A - B$ is the difference of the sets $A$ and $B$. The *symmetric difference* of the sets $A$ and $B$ is denoted by $A \oplus B$. We have

$$A \oplus B = (A - B) \cup (B - A).$$
Definition

Let $S$ be a set. A **sequence of length $n$ on $S$** is a mapping $s : \{1, \ldots, n\} \rightarrow S$. The set of sequences of length $n$ on $S$ is denoted by $\text{Seq}_n(S)$.

An **ordered pair on $S$** is a sequence of length 2 on $S$; a **singleton** is a sequence of length 1.

If $s$ is a sequence of length $n$ on $S$ and $s(i) = x_i$ for $1 \leq i \leq n$, we write $s = (x_1, \ldots, x_n)$. The elements $x_1, \ldots, x_n$ are the **components** of $s$.

The length of a sequence $s$ is denoted by $|s|$. 
Example

A sequence of natural numbers of length 6 is \( s = (6, 5, 2, 4, 9, 6) \). Note that in a sequence the same element of \( S \) may occur on multiple positions.
Counting Sequences

If $S$ is a finite set containing $m$ elements, then there are $m^n$ sequences of length $n$ for any $n \geq 1$. We extend the definition of sequences on $S$ be defining the *null sequence on $S$* as the sequence $\lambda$ that has no components, $\lambda = ()$. Note that there exists exactly one such sequence on $S$ and this is consistent with the fact that $m^0 = 1$ for every $m \geq 1$.

The *set of sequences of elements of $S$* is the set

$$\text{Seq}(S) = \bigcup \{\text{Seq}_n(S) \mid n \geq 0\}.$$
Operations with Sequences

If \( s = (s_1, s_2, \ldots, s_n) \) is a sequence in \( S \), we refer to the sequence \( \tilde{s} = (s_n, \ldots, s_2, s_1) \) as the **reversal** of the sequence \( s \). Clearly \( \tilde{\lambda} = \lambda \).

If \( s = (s_1, \ldots, s_n) \) and \( t = (t_1, \ldots, t_m) \) are two sequences on a set \( S \), their **concatenation** is the sequence \( st = (s_1, \ldots, s_n, t_1, \ldots, t_m) \). For the null sequence we define \( \lambda s = s \lambda = s \) for every \( s \in \text{Seq}(S) \). Note that \( |st| = |s| + |t| \) for all sequences \( s, t \in \text{Seq}(S) \).

Note that sequence concatenation is not a commutative operation in general.
Example

Let $s = (1, 2, 3), t = (4, 5)$. We have

$$st = (1, 2, 3, 4, 5) \text{ and } ts = (4, 5, 1, 2, 3),$$

so $st \neq ts$.

Sequence concatenation is an associative operation on $\text{Seq}(S)$, that is $(st)u = s(tu)$ for every $s, t, u \in \text{Seq}(S)$. 
Operations with Collections of Sets

Let $\mathcal{C} = \{ S_i \mid i \in I \}$ be a collection of sets. Its union is the set $U$ defined as

$$U = \bigcup_{i \in I} S_i.$$ 

Note that $\mathcal{C} \subseteq \mathcal{C}'$ implies $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{C}'$.

Unlike the union, the intersection is defined only for collections that consist of subsets of a set $S$.

If $\mathcal{C}$ is a collection of subsets of $S$, that is, if $\mathcal{C} \subseteq \mathcal{P}(S)$, the intersection of $\mathcal{C}$ is the set of all elements of $S$ that belong to every set of $\mathcal{C}$. The intersection of $\mathcal{C}$ is denoted by $\bigcap \mathcal{C}$. 

If $C$ and $C'$ are two collections of subsets of a set $S$ and $C \subseteq C'$, then $\bigcap C' \subseteq \bigcap C$. If $\emptyset$ is the empty collection of subsets of $S$, we define $\bigcap \emptyset = S$. 
Definition

A *closure system on the set* \( S \) is a collection \( \mathcal{K} \) of subsets of \( S \) such that for every collection of subsets \( \mathcal{C} \) such that \( \mathcal{C} \subseteq \mathcal{K} \) we have \( \bigcap \mathcal{C} \in \mathcal{K} \).

Note that if \( \mathcal{K} \) is a closure system on a set \( S \), then \( S \in \mathcal{K} \) because \( S \) is the intersection of the empty collection of subsets of \( \mathcal{K} \).
Definition

Let $\mathcal{K}$ be a closure system on a set $S$ and let $T$ be a subset of $S$. The **closure of $T$ relative to the closure system $\mathcal{K}$** is the set $\mathcal{K}(T) = \bigcap\{U \in \mathcal{K} \mid T \subseteq U\}$.

For every set $T$ the collection $\mathcal{C}_T = \{U \in \mathcal{K} \mid T \subseteq U\}$ is non-empty because it includes at least $S$. The set $\bigcap \mathcal{C}_T$ is denoted by $\mathcal{K}(T)$ and is referred to as the **closure** of $T$.

To emphasize that the closure of $T$ is computed relative to the closure system $\mathcal{K}$ we may denote this closure by $\mathcal{K}_\mathcal{K}(T)$. 
A subset \( E \) of \( \mathbb{R} \) is said to be symmetric if \( x \in E \) if and only if \( -x \in E \). Let \( \{ E_i \mid i \in I \} \) be a collection of symmetric subsets of \( \mathbb{R} \). It is easy to see that \( \bigcap \{ E_i \mid i \in I \} \) is a symmetric set. Note that \( \mathbb{R} \) itself is symmetric. Thus, the collection \( \mathcal{E} \) of symmetric subsets of \( \mathbb{R} \) is a closure system. For a subset \( T \) of \( \mathbb{R} \) the set \( K_\mathcal{E}(T) \) is the smallest symmetric set that includes \( T \).
Let $A$ and $B$ be two finite sets. It is easy to verify that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$ 

In this section we discuss a generalization of this Equality known as the *inclusion-exclusion principle*.

**Definition**

Let $S$ be a set and let $U$ be a subset of $S$. The *indicator* of $U$ is the function $I_U : S \rightarrow \{0, 1\}$ given by

$$I_U(x) = \begin{cases} 
1 & \text{if } x \in U, \\
0 & \text{if } x \notin U.
\end{cases}$$
If $U$ and $V$ are two subsets of a finite set $S$ such that $V \subseteq U$, then the function $I$ defined by $I(x) = I_U(x) - I_V(x)$ for $x \in S$ is an indicator function, namely the indicator function of the subset $U - V$ of $S$. 

Let $a$ and $b$ be two numbers that belong to the set $\{-1, 1\}$ such that the function $I_{ab}$ defined by

$$I_{ab}(x) = aI_U(x) + bI_V(x)$$

for $x \in S$ is the indicator function of a subset $W$ of the set $S$. We need to examine conditions under which $I_{ab}$ is the indicator function of a set.
Since $l_{ab}(x) \in \{0, 1\}$ and $l_{ab}(x) = al_U(x) + bl_V(x)$, the following cases are possible:

- If $a = b = 1$, then we have $U \cap V = \emptyset$; otherwise (that is, if $x \in U \cap V$) we would have $al_U(x) + bl_V(x) = 2$ and this would prevent $l_{ab}$ from being an indicator function. Clearly, in this case, $W = U \cup V$.
- If $a = 1$ and $b = -1$, we must have $l_V(x) \leq l_U(x)$ for every $x \in S$, which implies $V \subseteq U$. Thus, $W = U - V$.
- The case where $a = -1$ and $b = 1$ is similar to the previous case, and we have $W = V - U$.
- The case when $a = -1$ and $b = -1$ is possible only if $U = V = \emptyset$. In this case, $W = \emptyset$.

Note that in all these cases we have $|W| = a|U| + b|V|$. This observation is generalized by the next statement.
Theorem

Let $U_0, \ldots, U_{n-1}$ be $n$ subsets of a finite set $S$, where $n \geq 2$, and let $(a_0, \ldots, a_{n-1}) \in \text{Seq}_n(\{-1, 1\})$ be a sequence of $n$ numbers such that the function $I : S \rightarrow \{0, 1\}$ defined by

$$I(x) = a_0 I_{U_0}(x) + \cdots + a_{n-1} I_{U_{n-1}}(x)$$

for $x \in S$ is the indicator function of a subset $W$ of $S$. Then,

$$|W| = a_0 |U_0| + \cdots + a_{n-1} |U_{n-1}|.$$
Proof

If $W$ is a subset of $S$, then $\sum_{x \in S} l_W(x) = |W|$ because for each $x \in S$ its contribution to the sum $\sum_{x \in S} l_W(x)$ is equal to 1 if and only if $x \in W$. Therefore, if $l_W(x) = \sum_{i=0}^{n-1} a_i l_{U_i}(x)$ for $x \in S$, we have

\[
|W| = \sum_{x \in S} l_W(x) = \sum_{x \in S} \sum_{i=0}^{n-1} a_i l_{U_i}(x) = \sum_{i=0}^{n-1} \sum_{x \in S} a_i l_{U_i}(x) = \sum_{i=0}^{n-1} \sum_{x \in S} a_i l_{U_i}(x) = \sum_{i=0}^{n-1} a_i |U_i|.
\]
Corollary

(Principle of Inclusion-Exclusion) Let $A_0, \ldots, A_{n-1}$ be $n$ finite sets, where $n \geq 2$. We have

$$\left| \bigcup_{i=0}^{n-1} A_i \right| = \sum_{0 \leq i \leq n-1} |A_i| - \sum_{0 \leq i_1 < i_2 \leq n-1} |A_{i_1} \cap A_{i_2}| + \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots + (-1)^{n+1} |A_0 \cap \cdots \cap A_{n-1}|.$$
Proof

Suppose that \( A_i \subseteq S \) for \( 0 \leq i \leq n - 1 \), where \( S \) is a finite set. For \( x \in S \), we have \( x \notin A = \bigcup_{i=0}^{n-1} A_i \) if and only if \( x \notin A_i \) for \( 0 \leq i \leq n - 1 \). This is equivalent to writing

\[
1 - I_A(x) = (1 - I_{A_{i_0}}(x)) \cdots (1 - I_{A_{i_{n-1}}}(x))
\]

for every \( x \in S \). This equality is, in turn, equivalent to

\[
I_A(x)
\]

\[
= \sum_{i=0}^{n-1} I_{A_i}(x) - \sum_{0 \leq i_1 < i_2 \leq n-1} I_{A_{i_1}}(x)I_{A_{i_2}}(x)
\]

\[
+ \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} I_{A_{i_1}}(x)I_{A_{i_2}}(x)I_{A_{i_3}}(x) - \cdots + (-1)^{n+1} I_{A_0}(x) \cdots I_{A_{n-1}}(x)
\]

\[
= \sum_{i=0}^{n-1} I_{A_i}(x) - \sum_{0 \leq i_1 < i_2 \leq n-1} I_{A_{i_1} \cap A_{i_2}}(x)
\]

\[
+ \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} I_{A_{i_1} \cap A_{i_2} \cap A_{i_3}}(x) - \cdots + (-1)^{n+1} I_{A_0 \cap \cdots \cap A_{n-1}}(x).
\]
Corollary

Let $A_0, \ldots, A_{n-1}$ be $n$ finite sets, where $n \geq 2$, and let $S = \bigcup_{i=0}^{n-1} A_i$. We have

$$\left| \bigcap_{i=0}^{n-1} A_i \right| = |S| - \sum_{0 \leq i \leq n-1} |A_i| + \sum_{0 \leq i_1 < i_2 \leq n-1} |A_{i_1} \cap A_{i_2}|$$

$$- \sum_{0 \leq i_1 < i_2 < i_3 \leq n-1} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| + \cdots + (-1)^n |A_0 \cap \cdots \cap A_{n-1}|.$$

Proof: This follows immediately from the previous Corollary by observing that

$$\left| \bigcap_{i=0}^{n-1} A_i \right| = |S| - \left| \bigcup_{i=0}^{n-1} A_i \right|.$$
A relation on the set $S$ is a set of ordered pairs of $S$.

The set of relations on $S$ is denoted by $\text{rel}(S)$. Since relations on $S$ are sets of pairs on $S$ they can be involved in the usual set-theoretical operations: union, intersection, difference, etc. If $\rho, \sigma \in \text{rel}(S)$, the union, intersection, and difference of $\rho$ and $\sigma$ are denoted by $\rho \cup \sigma$, $\rho \cap \sigma$, and $\rho - \sigma$, respectively. Also, $\rho \subseteq \sigma$ denotes the inclusion of the set of pairs $\rho$ into the set of pairs $\sigma$. 
Two important relations on $S$ are the \textit{diagonal relation}

$$\iota_S = \{(x, x) \mid x \in S\},$$

and the \textit{total relation}

$$\theta_S = \{(x, y) \mid x, y \in S\}.$$

\begin{definition}
Let $\rho, \sigma \in \text{rel}(S)$. The \textit{product} of $\rho$ and $\sigma$ is the relation $\rho\sigma$ given by

$$\rho\sigma = \{(x, z) \in \text{Seq}_2(S) \mid (x, y) \in \rho \text{ and } (y, z) \in \sigma\}.$$ 
\end{definition}
A relation $\rho \in \text{rel}(S)$ is:

- **reflexive**, if $\iota_S \subseteq \rho$;
- **symmetric**, if $(x, y) \in \rho$ is equivalent to $(y, x) \in \rho$;
- **antisymmetric**, if $(x, y) \in \rho$ and $(y, x) \in \rho$ implies $x = y$;
- **transitive**, if $(x, y), (y, z) \in \rho$ implies $(x, z) \in \rho$,

for all $x, y, z \in S$.

If $\rho \in \text{rel}(S)$, the **inverse** of $\rho$ is the relation

$$\rho^{-1} = \{(y, x) \in S \times S \mid (x, y) \in \rho\}.$$
The $n^{\text{th}}$ **power of a relation** $\rho$, where $\rho \subseteq S \times S$ is defined inductively as

\[
\begin{align*}
\rho^0 &= \iota_S, \\
\rho^{n+1} &= \rho^n \rho
\end{align*}
\]

for $n \geq 0$. 

If $\rho$ is a relation on $S$, then $(x, x) \in \rho^0$ for every $x \in S$. An easy argument by induction on $n \in \mathbb{N}$ shows that $(x, y) \in \rho^n$ if and only if there exists a sequence $z = (z_0, z_1, \ldots, z_n)$ of length $n + 1$ such that $x = z_0$, $(z_i, z_{i+1}) \in \rho$ for $0 \leq i \leq n - 1$ and $z_n = y$.
Properties of relations can be expressed using the operations just introduced. For example, a relation $\rho$ on a set $S$ is symmetric if and only if $\rho^{-1} = \rho$; a relation $\rho$ is transitive if $\rho^2 \subseteq \rho$.

**Definition**

An *equivalence relation* on a set $S$ is a relation $\rho$, $\rho \subseteq S \times S$ that is reflexive, symmetric, and transitive. The set of equivalence relations on $S$ is denoted by $\text{EQ}(S)$. 
Example

Both $\nu_S$ and $\theta_S$ are equivalence relations on $S$; moreover, for any equivalence $\rho \in \text{EQ}(S)$ we have $\nu_S \subseteq \rho \subset \theta_S$. 
Example

Let $m$ be a positive integer. Define the relation $\equiv_m$ on $\mathbb{Z}$ as consisting of those pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ if $p - q = km$ for some $k \in \mathbb{Z}$. In other words, we have $(p, q) \in \equiv_m$ if $p - q$ is divisible by $m$.

Note that $(r, r) \in \equiv_m$ because $r - r = 0$ is divisible by $m$. If $p - q = km$ for some $k \in \mathbb{Z}$, then $q - p = (-k)m$, so $(p, q) \in \equiv_m$ implies $(q, p) \in \equiv_m$. Finally suppose that $(p, q) \in \equiv_m$ and $(q, s) \in \equiv_m$. Since $p - q = km$ and $q - s = hm$, we have $p - s = (k + h)m$, hence $(p, s) \in \equiv_m$. Thus, $\equiv_m$ is an equivalence relation on $\mathbb{Z}$.
Following common practice, for an equivalence \( \rho \) on a set \( S \) and for \( x, y \in S \) we write \( x \rho y \) for \( (x, y) \in \rho \).

**Definition**

Let \( \rho \) be an equivalence relation on a set \( S \). The *equivalence class* of an element \( x \) of \( S \) is the set

\[
[x]_\rho = \{ u \in S \mid (x, u) \in \rho \}.
\]

By the reflexivity of \( \rho \), \((x, x) \in \rho \) for every \( x \in S \). Thus, \( x \in [x]_\rho \), hence each equivalence class is non-empty.
Lemma

Let $\rho$ be an equivalence relation on a set $S$. We have $y \in [x]_\rho$ if and only if $[y]_\rho = [x]_\rho$. 
Proof

Suppose that \( y \in [x]_\rho \) and that \( u \in [y]_\rho \). Then, we have \((x, y) \in \rho\) and \((y, u) \in \rho\). By transitivity, \((x, u) \in \rho\), that is, \( u \in [x]_\rho \), which implies \([y]_\rho \subseteq [x]_\rho\).

If \( v \in [x]_\rho \), then \((x, v) \in \rho\). Since \((x, y) \in \rho\), by the symmetry and transitivity of \( \rho \) we obtain \((y, v) \in \rho\), hence \( v \in [y]_\rho \), so \([x]_\rho \subseteq [y]_\rho\). This implies \([x]_\rho = [y]_\rho\).

Conversely, if \([y]_\rho = [x]_\rho\), we have \( y \in [x]_\rho \) because \( y \in [y]_\rho \).
Theorem

Let $\rho$ be an equivalence relation on a set $S$. If $[x]_\rho \neq [y]_\rho$, then $[x]_\rho \cap [y]_\rho = \emptyset$.

Proof.

Let $x, y \in S$ be such that $[x]_\rho \neq [y]_\rho$ and suppose that $z \in [x]_\rho \cap [y]_\rho$. Since $z \in [x]_\rho$ we have $[z]_\rho = [x]_\rho$; similarly, since $z \in [y]_\rho$ we have $[z]_\rho = [y]_\rho$, which means that $[x]_\rho = [y]_\rho$. This contradicts the hypothesis.
Definition

Let $S$ be a non-empty set. A \textit{partition} on $S$ is a non-empty collection $\pi = \{B_i \mid i \in I\}$ such that
- $B_i \neq \emptyset$ for $i \in I$;
- $i, j \in I$ and $i \neq j$ implies $B_i \cap B_j = \emptyset$;
- $\bigcup_{i \in I} B_i = S$.

The sets $B_i$ are the \textit{blocks} of the partition $\pi$.

The set of partitions of a set $S$ is denoted by $\PART(S)$; the set of partitions of $S$ that have $k$ blocks, where $1 \leq k \leq |S|$ is denoted by $\PART_k(S)$.

The partitions in $\PART_2(S)$ are referred to as \textit{bipartitions}.

Clearly, $\PART(S) = \bigcup_{k=1}^{|S|} \PART_k(S)$. 

\begin{center}
\includegraphics[width=\textwidth]{image.png}
\end{center}
Example

The partition of a set $S$ that consists of all singletons $\{x\}$, where $x \in S$ is denoted by $\alpha_S$; the partition of $S$ that contains one block, namely $S$, is denoted by $\omega_S$. We have $\text{PART}_{|S|} = \{\alpha_S\}$ and $\text{PART}_1(S) = \{\omega_S\}$. 
Example

Let $\rho$ be an equivalence relation on a set $S$. The set of equivalence classes of $\rho$ is a partition of the set $S$. Indeed, we saw that $S = \bigcup_{x \in S} [x]_{\rho}$, no equivalence class is empty and, as we saw, any two equivalence classes are disjoint.

The set of equivalence classes of an equivalence relation is known as the quotient set of $S$ by $\rho$ and is denoted by $S/\rho$. The partition generated by the equivalence relation is also denoted by $\pi_{\rho}$. 
Let $m \in \mathbb{P}$ and let $B_i$ be the set of all members $n$ of $\mathbb{P}$ such that the remainder of the division of $n$ by $m$ equals $i$, where $0 \leq i \leq m - 1$. It is immediate that the collection \{\(B_0, B_1, \ldots, B_{m-1}\)\} is a partition of the set $\mathbb{P}$. For instance, if $m = 3$, we have $B_0 = \{3, 6, 9, 12 \ldots\}$, $B_1 = \{1, 4, 7, 10, \ldots\}$, and $B_2 = \{2, 5, 8, 11, \ldots\}$.
Theorem

Let $\pi = \{B_i \mid i \in I\}$ be a partition of the set $S$. The relation $\rho_\pi$ defined by

$$\rho_\pi = \{(x, y) \in S \times S \mid \{x, y\} \subseteq B_i \in \pi\}$$

is an equivalence on $S$. 
Proof

Each $x$ belongs to a block $B_i$ of $\pi$, so $(x, x) \in \rho_\pi$ for every $x \in S$, which means that $\rho_\pi$ is reflexive.

If $(x, y) \in \rho_\pi$, then $\{x, y\} \subseteq B_i$, which obviously implies $(y, x) \in \rho_\pi$, so $\rho_\pi$ is symmetric.

Finally, if $(x, y) \in \rho_\pi$ and $(y, z) \in \rho_\pi$, there exist $B_i, B_j \in \pi$ such that $\{x, y\} \subseteq B_i$ and $\{y, z\} \subseteq B_j$. Thus, $B_i \cap B_j \neq \emptyset$ (because both contain $y$), which implies $B_i = B_j$. Therefore, $\{x, z\} \subseteq B_i = B_j$, hence $(x, z) \in \rho_\pi$, which allows us to conclude that $\rho_\pi$ is an equivalence relation.
Corollary

Let $\pi \in \text{PART}(S)$ and let $\rho \in \text{EQ}(S)$ $\rho = \rho \pi \rho$ and $\pi = \pi \rho \pi$.

Proof.

The equalities follow easily from the definitions of $\pi \rho$ and $\rho \pi$. □
Example

Note that $\pi_S = \alpha_S$, $\pi_S = \omega_S$ and $\rho_S = \nu_S$, $\rho_S = \theta_S$.

We write $x \equiv y(\pi)$ to denote that $(x, y) \in \rho_{\pi}$.
Denote by $(x)_n$ the $n$-degree polynomial

$$(x)_n = x(x - 1) \cdots (x - n + 1).$$

The coefficients of this polynomial

$$(x)_n = s(n, n)x^n + s(n, n-1)x^{n-1} + \cdots + s(n, i)x^i + \cdots + s(n, 0)$$

are known as the Stirling numbers of the first kind.
Theorem

We have:

\[ s(n, 0) = 0, \]
\[ s(n, n) = 1, \]
\[ s(n + 1, k) = s(n, k - 1) - ns(n, k). \]

Proof.

The verification of the first two equalities is immediate. The third equality follows by observing that \((x)_{n+1} = (x)_n(x - n)\) and seeking the coefficient of \(x^k\) on both sides.
Let $S$ be a set having $n$ elements. We are interested in the number of partitions of $S$ that have $k$ blocks. We begin by counting the number of onto functions of the form $f : A \rightarrow B$, where $|A| = n$, $|B| = k$, and $n \geq k$.

**Lemma**

Let $A$ and $B$ be two sets, where $|A| = n$, $|B| = k$, and $n \geq k$. The number of surjective functions from $A$ to $B$ is given by

$$
\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k - j)^n.
$$
Proof

There are $k^n$ functions of the form $f : A \rightarrow B$. We begin by determining the number of functions that are not surjective. Suppose that $B = \{b_1, \ldots, b_k\}$, and let $F_j = \{f : A \rightarrow B \mid b_j \notin f(A)\}$ for $1 \leq j \leq k$. A function is not surjective if it belongs to one of the sets $F_j$. Thus, we need to evaluate $|\bigcup_{j=1}^k F_j|$. By using the inclusion-exclusion principle, we can write:

$$\left|\bigcup_{j=1}^k F_j\right| = \sum_{j_1=1}^k |F_{j_1}| - \sum_{j_1,j_2=1}^k |F_{j_1} \cap F_{j_2}|$$

$$+ \sum_{j_1,j_2,j_3=1}^k |F_{j_1} \cap F_{j_2} \cap F_{j_3}| - \cdots - +(-1)^k |F_1 \cap F_2 \cap \cdots \cap F_k|.$$
Proof (cont’d)

Note that the set \(|F_{j_1} \cap F_{j_2} \cap \cdots \cap F_{j_p}|\) is actually the set of functions defined on \(A\) with values in the set \(B - \{y_{j_1}, y_{j_2}, \ldots, y_{j_p}\}\), and there are \((k - p)^n\) such functions. Since there are \(\binom{k}{p}\) choices for the set \(\{j_1, j_2, \ldots, j_p\}\), it follows that there are

\[
\binom{k}{1}(k - 1)^n - \binom{k}{2}(k - 2)^n + \binom{k}{3}(k - 3)^n - \cdots + (-1)^k \binom{k}{k-1}
\]

functions that are not surjective.
Proof (cont’d)

Thus, we can conclude that there are

\[ \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k - j)^n \]

\[ = k^n - \binom{k}{1} (k - 1)^n + \binom{k}{2} (k - 2)^n - \cdots + (-1)^{k-1} \binom{k}{k-1} \]

surjective functions from \( A \) to \( B \).
Theorem

The number of partitions of a set $S$ that have $k$ blocks ($k \leq n$) is given by

$$
\frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k - j)^n.
$$
Proof

Note that there are $k!$ distinct onto functions that have the same kernel partition. Indeed, given a surjective function $f : A \rightarrow B$, one can obtain a function $g$ that has the same partition as $f$ by defining $g(a) = p(f(a))$, where $p$ is a permutation of the set $B$, that is, a bijection $p : B \rightarrow B$. Since there are $k!$ such bijections, it follows that the number of partitions is

$$\frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^n.$$
The numbers $S(n, k)$ defined by

$$S(n, k) = \begin{cases} \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k - j)^n & \text{if } n \geq k > 0, \\ 1 & \text{if } n = k = 0, \\ 0 & \text{in other cases.} \end{cases}$$

for $n, k \in \mathbb{N}$ and are known as the \textit{Stirling numbers of the second kind}. 
Example

Note that $S(n, 1) = 1$ and $S(n, n) = 1$ because only one partition of a set with $n$ elements, $\omega_S$, has one block, and only one partition of a set with $n$ elements, $\alpha_S$ has $n$ blocks which are singletons.

The number of partitions of a 4-element set having two blocks is

$$S(4, 2) = \frac{1}{2!} \sum_{j=0}^{1} \binom{2}{j} (2 - j)^4$$

$$= \frac{1}{2!} \left( \binom{2}{0} \cdot 2^4 - \binom{2}{1} \cdot 1^4 \right) = 7.$$

Namely, these partitions are:

$$\{\{1\}, \{2, 3, 4\}\}, \{\{2\}, \{1, 3, 4\}\}, \{\{3\}, \{1, 2, 4\}\}, \{\{4\}, \{1, 2, 3\}\},$$

$$\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}.$$
We claim that

\[ S(n, k) = kS(n - 1, k) + S(n - 1, k - 1). \]

Indeed, observe that a partition \( \pi \) of the set \( \{1, \ldots, n - 1\} \) can be transformed into a partition of \( \{1, \ldots, n\} \) be adjoining \( n \) to one of the blocks of \( \pi \) or by increasing the number of blocks by 1 and making \( \{n\} \) a block.
Theorem

For every $n \geq 1$ we have $m^n = \sum_{j=1}^{n} S(n,j)(m)_j$. 

Proof

Let $A$ and $B$ be two finite sets such that $|A| = n$ and $|B| = m$. There are $m^n$ functions $f : A \rightarrow B$. These functions can be classified depending on the size of their range $f(A)$. If $g : A \rightarrow B$ is a function such that $|g(A)| = j$, then $g$ can be regarded as a surjection from $A$ to $g(A)$. Since there are $j!S(n,j)$ such surjective functions and there are $\binom{m}{j}$ subsets of $B$ that have $j$ elements, we can write

$$m^n = \sum_{j=1}^{m} \binom{m}{j} j!S(n,j)$$

$$= \sum_{j=1}^{m} m(m-1) \cdots (m-j+1) S(n,j) = \sum_{j=1}^{m} (m)_n S(n,j)$$

for every $m \geq 1$. 

The *Bell number* $B_n$ is the total number of partitions of a set of $n$ objects, that is,

$$B_n = \sum_{k=1}^{n} S(n, k).$$

**Example**

For $n = 4$, we have shown that there exist 7 partitions having two blocks, one partition with one block and one partition with 4 blocks. It is easy to see that there are 6 partitions with 3 blocks, so $B_4 = 1 + 7 + 6 + 1 = 15$.

The first 10 values of the Bell numbers are given below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
<td>877</td>
<td>4140</td>
<td>21147</td>
<td>115975</td>
</tr>
</tbody>
</table>
Definition

A relation $\rho$ is a *partial order* on a set $S$ if $\rho$ is reflexive, antisymmetric and transitive.

A *partially ordered set* (or, a *poset*) is a pair $(S, \rho)$, where $\rho$ is a partial order on $S$.

In general, we denote partial orders using the symbol “$\leq$” or similar symbols; furthermore, instead of writing $(x, y) \in \leq$, we write $x \leq y$. 
Example

Let $T$ be a set. The set of subsets of $T$, $\mathcal{P}(T)$ equipped with the set inclusion “$\subseteq$” yields the poset $((\mathcal{P}(T), \subseteq)$.
Example

The pair \((\mathbb{P}, |)\), where “|” is the divisibility relation is a poset defined by \(p|q\) if there exists \(k \in \mathbb{P}\) such that \(q = pk\). Indeed, we have \(p|p\) for every \(p \in \mathbb{P}\), so “|” is reflexive. If \(p|q\) and \(q|p\), we have \(q = pk\) and \(p = qh\), hence \(hk = 1\) which implies \(h = k = 1\). Thus, \(p = q\), which shows that “|” is antisymmetric. Finally, if \(p|q\) and \(q|r\) we have \(q = pk\) and \(r = qh\) for some \(k, h \in \mathbb{P}\). Thus, \(r = pkh\), so \(p|r\).
If \((S, \rho)\) is a poset and \(T \subseteq S\), it is easy to see that the relation 
\[\rho_T = \rho \cap (T \times T)\] is itself a partial order; we will refer to it as the \textit{trace of }\((S, \rho)\) \textit{on }\(T\). Often, we will use the same symbol \(\rho\) instead of \(\rho_T\) to denote the partial order on \(T\).

**Example**

Let \(S \subseteq \mathbb{P}\) be the set \(\{1, 2, 3, 4, 5, 6\}\). The trace of \(\mathbb{P}\) on \(S\) consists of the pairs:

\[(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6).\]
Definition

A **totally ordered set** is a pair $(S, \rho)$, where $\rho$ is a partial order with the additional property that for all $x, y \in S$ we have either $(x, y) \in \rho$, or $(y, x) \in \rho$. The relation $\rho$ is referred to as a **total order**.

Example

The real numbers $\mathbb{R}$ equipped with the standard less-than-or-equal relation $\leq$ is a totally ordered set.
Definition

A sequence $x \in \text{Seq}(S)$ is a \textit{subsequence of a sequence} $y \in \text{Seq}(S)$, if $y = u xv$ for some sequences $u, v \in \text{Seq}(S)$. This is denoted by $x \sqsubseteq y$.

Example

Let $S = \{0, 1\}$. The sequence $y = 1011$ is a subsequence of $x = 010110110101100$.

The relation “$\sqsubseteq$” is a partial order on $\text{Seq}(S)$.
Definition

Let $(P, \leq)$ be a poset. An element $y$ covers an element $x$ of $P$ if $x \leq y$ and there is no $z \in P$, $z \neq x$ and $z \neq y$ such that $x \leq z \leq y$. We denote the fact that $y$ covers $x$ by $x \prec y$. 

Prof. Dan A. Simovici
CS742: Topics in Algorithms
Set-Theoretical Preliminaries
Example

Let $(\mathbb{P}, |)$ be the poset of positive numbers equipped with the divisibility relation. We have $p \prec y$ if $x$ is none of the largest divisors of $y$. For example, we have $6 \prec 12$ because there is no number $z$ distinct from 6 and 12 such that $6 | z$ and $z | 12$. Note that $3 | 12$ but $3 \not\prec 12$.

Finite posets can be represented graphically using **Hasse diagrams**. Each element is represented by a dot. If $x, y$ are elements of a poset $(P, \leq)$ and $x \prec y$, then the dot representing $y$ is placed at a greater height than $x$ and a link between the dots is drawn.
Example

The Hasse diagram of the poset \((\{1, \ldots, 12\}, \mid)\) is given below:
Definition

Let \((S, \leq)\) be a poset and let \(T\) be a subset of \(S\. The \textit{set of upper bounds} of \(T\) is the set

\[
T^s = \{ y \in S \mid \text{for all } x \in T \text{ we have } x \leq y \}.
\]

The \textit{set of lower bounds} of \(T\) is the set

\[
T^i = \{ y \in S \mid \text{for all } x \in T \text{ we have } y \leq x \}.
\]
If $T_1, T_2$ are two subsets of $S$, $T_1 \subseteq T_2$ implies $T_2^s \subseteq T_1^s$ and $T_2^i \subseteq T_1^i$.  

**Theorem**

Let $(S, \leq)$ be a poset and let $T$ be a subset of $S$. The sets $T \cap T^s$ and $T \cap T^i$ contain at most one element of $S$.

**Proof.**

Suppose that $x, y \in T \cap T^s$. Since $x \in T$ and $y \in T^s$, it follows that $x \leq y$. On other hand, since $x \in T^s$ and $y \in T$ we have $y \leq x$. Therefore, $x = y$, which implies that the set $T \cap T^s$ contains at most one element. The argument for $T \cap T^i$ is similar.
Definition

Let \((S, \leq)\) be a poset and let \(T\) be a subset of \(S\). If \(T \cap T^s = \{u\}\), then \(u\) is the largest element of set \(T\).

If \(T \cap T^i = \{v\}\), then \(v\) is the least element of set \(T\).

Example

Not every subset of a poset has a least or a greatest element. The subset \(\{1, 2, 3, 6\}\) of the poset \((\{1, \ldots, 12\}, \mid)\) considered before has 1 as its least element and 6 as the largest element. In contrast, the set \(\{4, 5, 6\}\) has neither a least nor a largest element.
If $T$ is a subset of a poset $(S, \leq)$ we will consider the sets $(T^s)^i$ and $(T^i)^s$ denoted by $T^{si}$ and $T^{is}$, respectively.

Observe that the set $T^s \cap T^{si} = T^s \cap (T^s)^i$ may contain at most one element, by a previous observation applied to the set $T^s$. Similarly, the set $T^i \cap T^{is}$ may contain at most one element.
Definition

Let \((S, \leq)\) be a poset and let \(T\) be a subset of \(S\). If \(T^s \cap T^{si} = \{u\}\), \(u\) is the *supremum* of the set \(T\).
If \(T^i \cap T^{is} = \{v\}\), \(v\) is the *infimum* of \(T\).

The supremum and infimum of a set \(T\) (if they exist) are unique and are denoted by \(\text{sup } T\) and \(\text{inf } T\), respectively.
Example

In the poset \((\mathcal{P}(T), \subseteq)\) introduced in before, for every \(C \in \mathcal{P}(X)\) we have

\[
\inf C = \bigcap C \quad \text{and} \quad \sup C = \bigcup C.
\]
Example

In the poset $(\mathbb{P}, |)$, we have

$$\inf\{p, q\} = \gcd(p, q) \text{ and } \sup\{p, q\} = \text{lcm}(p, q),$$

where $\gcd(p, q)$ is the greatest common divisor of $p$ and $q$, and $\text{lcm}(p, q)$ is the least common multiple of $p$ and $q$. 
Definition

A poset \((S, \leq)\) is a **lattice** if for every two elements \(x, y \in S\) there exist inf\(\{x, y\}\) and sup\(\{x, y\}\). If \((S, \leq)\) is a lattice we use the notations

\[
x \land y = \inf\{x, y\} \quad \text{and} \quad x \lor y = \sup\{x, y\}.
\]

The element \(x \land y\) is referred to as the **meet** of \(x\) and \(y\); \(x \lor y\) is the **join** of \(x\) and \(y\).

A poset \((S, \leq)\) is a **complete lattice** if for every \(X \in \mathcal{P}(S)\) there exist inf\(X\) and sup\(X\).
Example

- $(\mathcal{P}, |)$ is a lattice;
- $(\mathcal{P}(T), \subseteq)$ is a complete lattice.
Note that if \( (S, \leq) \) is a complete lattice and \( S \neq \emptyset \), then this poset has a least element \( 0_S = \inf S \), and a greatest element \( 1_S = \sup S \).

**Theorem**

Let \( (S, \leq) \) be a complete lattice and let \( W \) be a subset of \( S \) such that \( 1_S \in W \) and \( T \subseteq W \) implies that \( \inf T \) in \( S \) belongs to \( W \). Then \( W \) is a complete lattice.
Proof

For every nonvoid subset $T$ of $W$, $\inf T \in W$ and is the infimum of $T$ in $S$. Let $U$ be a subset of $W$ defined as $U = T^s$. We have $U \neq \emptyset$ because $1_S \in W$. Then, $\inf U \in W$ is also an upper bound of $T$, and is actually the least upper bound of $U$. Thus, $(W, \leq)$ is a complete lattice.
Let $\mathcal{K}$ be a closure system on a set $S$. The subsets of $S$ in $\mathcal{K}$ form a complete lattice in which $\inf \mathcal{C} = \bigcap \mathcal{C}$ and $\sup \mathcal{C} = \bigcap \{ T \in \mathcal{P} \mid C \subseteq T \text{ for every } C \in \mathcal{K} \}$. 
Let $\pi, \sigma$ be two partitions of $S$. We write $\pi \leq \sigma$ if each block $B$ of $\pi$ is included in a block $C$ of $\sigma$.

**Theorem**

The pair $(\text{PART}(S), \leq)$ is a partially ordered set.
Proof

The relation “≤” is obviously reflexive. Suppose that we have both \( \pi \leq \sigma \) and \( \sigma \leq \pi \). Then, a block \( B \) of \( \pi \) is included in a block \( C \) of \( \sigma \), and \( C \), in turn, is included in a block \( B' \) of \( C \). Thus, \( B \subseteq C \subseteq B' \), which implies \( B = C = B' \) because no block of \( \pi \) can be included into another block. Thus, \( \pi \subseteq \sigma \). In the same manner, starting from a block \( C \) of \( \sigma \) we can show that \( \sigma \subseteq \pi \), so \( \pi = \sigma \). This shows that the relation “≤” is antisymmetric. It is immediate that “≤” is transitive.
Let $\pi, \sigma \in \text{PART}(S)$ be two partitions, $\pi = \{B_i \mid i \in I\}$ and $\sigma = \{C_j \mid j \in J\}$. We have $\pi \leq \sigma$ if and only if for each $j \in J$ there exists a subset $l_j$ of $I$ such that $C_j = \bigcup \{B_i \mid i \in l_j\}$.

Suppose that $\pi \leq \sigma$ and let $C \in \sigma$. Suppose that $B \cap C \neq \emptyset$. Since each block $B$ of $\pi$ is included in a block $C'$ of $\sigma$ we must have $C' = C$ because, otherwise $C'$ and $C$ would have a non-empty intersection. Thus, if a block $B$ of $\pi$ has a non-empty intersection with a block $C$ of $\sigma$ we must have $B \subseteq C$. This implies that a block $C$ of $\sigma$ is a union of block of $\pi$. The converse implication is immediate.
Example

If $\pi \in \text{PART}(S)$ we have $\alpha_S \leq \pi \leq \omega_S$. Thus, $\alpha_S$ is the smallest element of $\langle \text{PART}(S), \leq \rangle$ and $\omega_S$ is its largest element.
Definition

Let $\pi, \sigma$ be two partitions of a set $S$. The partition $\sigma$ covers $\pi$ if $\pi < \sigma$ and there is no partition $\tau \in PART(S)$ such that $\pi < \tau < \sigma$. 
Let $\pi, \sigma$ be two partitions of a set $S$. The partition $\sigma$ covers $\pi$ if and only if there exists a block $C$ of $\sigma$ that is the union of two blocks $B$ and $B'$ of $\pi$, and every other block of $\sigma$ that is distinct of $C$ is a block of $\pi$. 
Suppose that $\sigma$ is a partition that covers the partition $\pi$. Since $\pi \leq \sigma$, every block of $\sigma$ is a union of blocks of $\pi$. Suppose that there exists a block $E$ of $\sigma$ that is the union of more than two blocks of $\pi$; that is, $E = \bigcup\{B_i \mid i \in I\}$, where $|I| \geq 3$, and let $B_{i_1}, B_{i_2}, B_{i_3}$ be three blocks of $\pi$ included in $E$. Consider the partitions

$$\sigma_1 = \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1}, B_{i_2}, B_{i_3}\},$$

$$\sigma_2 = \{C \in \sigma \mid C \neq E\} \cup \{B_{i_1} \cup B_{i_2}, B_{i_3}\}.$$

It is easy to see that $\pi \leq \sigma_1 < \sigma_2 < \sigma$, which contradicts the fact that $\sigma$ covers $\pi$. Thus, each block of $\sigma$ is the union of at most two blocks of $\pi$. 

Proof
Proof (cont’d)

Suppose that $\sigma$ contains two blocks $C'$ and $C''$ that are unions of two blocks of $\pi$, namely $C' = B_{i_0} \cup B_{i_1}$ and $C'' = B_{i_2} \cup B_{i_3}$. Define the partitions

\[
\sigma' = \{ C \in \sigma \mid C \notin \{C', C''\}\} \cup \{C', B_{i_2}, B_{i_3}\},
\]
\[
\sigma'' = \{ C \in \sigma \mid C \notin \{C', C''\}\} \cup \{B_{i_1}, B_{i_2}, C''\}.
\]

Since $\pi < \sigma', \sigma'' < \sigma$, this contradicts the fact that $\sigma$ covers $\pi$. Thus, we obtain the conclusion of the theorem.
Example

The Hasse diagram of \((PART(\{1, 2, 3\}), \leq)\) is given below:

\[
\begin{align*}
\{\{1, 2, 3\}\} \\
\{\{1, 2\}, \{3\}\} & \quad \{\{1\}, \{2, 3\}\} & \quad \{\{1, 3\}, \{2\}\} \\
\{\{1\}, \{2\}, \{3\}\}
\end{align*}
\]
Theorem

The posets \((EQS(S), \subseteq)\) and \((PART(S), \subseteq)\) are isomorphic.

Let \(f : EQS(S) \rightarrow PART(S)\) be the mapping defined by \(f(\rho) = S/\rho\). We need to show that \(f\) is a monotonic bijective mapping and that its inverse mapping \(f^{-1}\) is also monotonic.

The bijectivity of \(f\) follows immediately from the remarks that precede the theorem. Let \(\rho_0, \rho_1\) be two equivalences such that \(\rho_0 \subseteq \rho_1\) and let \(S/\rho_0 = \{B_i \mid i \in I\}\), \(S/\rho_1 = \{C_j \mid j \in J\}\). Let \(B_i\) be a block in \(S/\rho_0\) and assume that \(B_i = [x]_{\rho_0}\). We have \(y \in B_i\) if and only if \((x, y) \in \rho_0\), so \((x, y) \in \rho_1\). Therefore, \(y \in [x]_{\rho_1}\), which shows that every block \(B \in S/\rho_0\) is included in a block \(C \in \rho_1\). This shows that \(f(\rho_0) \subseteq f(\rho_1)\), so \(f\) is indeed monotonic.
Let \( \{\rho_i \mid i \in I\} \subseteq \text{EQS}(S) \) be a collection of equivalences. Then,

\[
\inf \{\rho_i \mid i \in I\} = \bigcap_{i \in I} \rho_i.
\]

**Definition**

Let \( S \) be a set and let \( \rho, \tau \in \text{EQS}(S) \). A \((\rho, \tau)\)-alternating sequence that joins \( x \) to \( y \) is a sequence \((s_0, s_1, \ldots, s_n)\) such that \( x = s_0, y = s_n \), \((s_i, s_{i+1}) \in \rho\) for every even \( i \) and \((s_i, s_{i+1}) \in \tau\) for every odd \( i \), where \( 0 \leq i \leq n - 1 \).
Lemma

Let $S$ be a set and let $\rho, \tau \in EQS(S)$. If $s$ and $z$ are two $(\rho, \tau)$-alternating sequences joining $x$ to $y$ and $y$ to $z$, respectively, then there exists a $(\rho, \tau)$-alternating sequence that joins $x$ to $z$. 
Proof

Let \((s_0, \ldots, s_n)\) be a \((\rho, \tau)\)-alternating sequences joining \(x\) to \(y\) and \((w_0, \ldots, w_m)\) a \((\rho, \tau)\)-alternating sequences joining \(y\) to \(z\), where \(x = s_0\), \(s_n = w_0 = y\) and \(w_m = z\). If \((s_{n-1}, s_n) \in \tau\), then the sequence \((s_0, \ldots, s_n, w_1, \ldots, w_m)\) is a \((\rho, \tau)\)-alternating sequence joining \(x\) to \(z\). Otherwise, that is, if \((s_{n-1}, s_n) \in \rho\), then taking into account the reflexivity of \(\tau\) we have \((s_n, w_0) = (s_n, s_n) \in \tau\). In this case, \((s_0, \ldots, s_n, s_n, w_1, \ldots, w_m)\) is a \((\rho, \tau)\)-alternating sequence joining \(x\) to \(z\).
Theorem

Let $S$ be a set and let $\rho, \tau \in EQS(S)$. If $\xi$ is the relation that consists of all pairs $(x, y) \in S \times S$ that can be joined by a $(\rho, \tau)$-alternating sequence, then $\xi = \text{sup}\{\rho, \tau\}$. 
It is easy to verify that $\xi$ is indeed an equivalence relation. Note that we have both $\rho \subseteq \xi$ and $\tau \subseteq \xi$. Indeed, if $(x, y) \in \rho$, then $(x, y, y)$ is a $(\rho, \tau)$-alternating sequence joining $x$ to $y$. If $(x, y) \in \tau$, then $(x, x, y)$ is the needed alternating sequence.

Let $\zeta \in EQS(S)$ such that $\rho \subseteq \zeta$ and $\tau \subseteq \zeta$. If $(x, y) \in \xi$, and $(s_0, s_1, \ldots, s_n)$ is a $(\rho, \tau)$-alternating sequence such that $x = s_0$, $y = s_n$, then each pair $(s_i, s_{i+1})$ belongs to $\zeta$. By the transitivity property, $(x, y) \in \zeta$, so $\xi \subseteq \zeta$. This implies that $\xi = \sup\{\rho, \tau\}$. 
If \( \pi, \sigma \in \text{PART}(S) \) both the infimum and the supremum of the set \( \{\pi, \sigma\} \) exist and their description follows from the corresponding results that refer to the equivalence relations. Namely, if \( \pi, \sigma \in \text{PART}(S) \), where
\[
\pi = \{B_i \mid i \in I\} \quad \text{and} \quad \sigma = \{C_j \mid j \in J\},
\]
the partition \( \inf\{\pi, \sigma\} \) exists and is given by
\[
\inf\{\pi, \sigma\} = \{B_i \cap C_j \mid i \in I, j \in J \text{ and } B_i \cap C_j \neq \emptyset\}.
\]
The partition \( \inf\{\pi, \sigma\} \) will be denoted by \( \pi \land \sigma \).
A block of the partition $\sup\{\pi, \sigma\}$, denoted by $\pi \vee \sigma$, is an equivalence class of the equivalence $\theta = \sup\{\rho_\pi \wedge \rho_\sigma\}$. We have $y \in [x]_\theta$ if there exists a sequence $(s_0, \ldots, s_n) \in \textbf{Seq}(S)$ such that $x = s_0, s_n = y$ and successive sets $\{s_i, s_{i+1}\}$ are included, alternatively, in a block of $\pi$ or in a block of $\sigma$. 