Let $F$ be the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. An $F$-linear space is a set $L$ on which two operations are defined: the addition $x + y$ of elements $x$ and $y$ of $L$ and the multiplication of an element $x$ of $L$ with a member $a$ of $F$, denoted by $ax$, such that the following conditions are satisfied:

I. Additive Conditions:
- addition is associative, that is, $x + (y + z) = (x + y) + z$;
- addition is commutative, that is, $x + y = y + x$;
- for every $x \in L$ there is an element $(-x)$ in $L$ such that $x + (-x) = 0_L$. 


II. Multiplicative Conditions:
- $L$ contains an element $0_L$ such that $0x = 0_L$;
- $(a + b)x = ax + bx$;
- $a(x + y) = ax + ay$;
- $(ab)x = a(bx)$;
- $1x = x$

for every $a, b \in F$ and $x, y \in L$. 

The elements of the field $\mathbb{F}$ are referred to as *scalars* while the elements of $L$ are referred to as *vectors*.

If the field $\mathbb{F}$ is irrelevant, or it is clearly designated from the context we refer to an $\mathbb{F}$-linear space just as a linear space. On another hand if $\mathbb{F}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ we designate an $\mathbb{R}$-linear space as a *real linear space* and a $\mathbb{C}$-linear space as a *complex linear space*. 
Example

If $F$ is a field, then the one-element linear space $L = \{0_L\}$, where $a0_L = 0_L$ for every $a \in F$ is the zero $F$-linear space, or, for short, the zero linear space.

The field $F$ itself is an $F$-linear space, where the Abelian group is $(F, \{0, +, -\})$ and scalar multiplication coincides with the scalar multiplication of $F$.

Note that the zero $F$-linear space is the smallest linear space.
Example

The set of all sequences of real numbers, $\text{Seq}(\mathbb{R})$ is a real linear space, where the sum of two sequences $\mathbf{x} = (x_0, x_1, \ldots)$ and $\mathbf{y} = (y_0, y_1, \ldots)$ is the sequence $\mathbf{x} + \mathbf{y}$ defined by $\mathbf{x} + \mathbf{y} = (x_0 + y_0, x_1 + y_1, \ldots)$ and the multiplication of $\mathbf{x}$ by a scalar $a$ is $a\mathbf{x} = (ax_0, ax_1, \ldots)$.

A related real linear space is the set $\text{Seq}_n(\mathbb{R})$ of all sequences of real numbers having length $n$, where the sum and the scalar multiplications are defined in a similar manner. Namely, if $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$, the sequence $\mathbf{x} + \mathbf{y}$ is defined by $\mathbf{x} + \mathbf{y} = (x_0 + y_0, x_1 + y_1, \ldots, x_{n-1} + y_{n-1})$ and the multiplication of $\mathbf{x}$ by a scalar $a$ is $a\mathbf{x} = (ax_0, ax_1, \ldots, ax_{n-1})$. This linear space is denoted by $\mathbb{R}^n$ and its zero element is denoted by $0_n$. 
Example

If the real field $\mathbb{R}$ is replaced by the complex field $\mathbb{C}$, we obtain the linear space $\text{Seq}(\mathbb{C})$ of all sequences of complex numbers. Similarly, we have the complex linear space $\mathbb{C}^n$ which consists of all sequences of length $n$ of complex numbers.
Example

Let $L$ be an $F$-linear space and let $S$ be a non-empty set. The set $L^S$ that consists of all functions of the form $f : S \rightarrow L$ is an $F$-linear space. The addition of functions is defined by

$$(f + g)(s) = f(s) + g(s),$$

while the multiplication by a scalar is given by $(af)(s) = af(s)$, for $s \in S$ and $a \in F$. 
Example

Let $\mathbb{R}[x]$ be the set of polynomials of variable $x$ with coefficients in $\mathbb{R}$. For example, $p \in \mathbb{R}[x]$, where

$$p(x) = 3x^7 - 5x^3 + x - 6.$$ 

The sum of two polynomials $p, q \in \mathbb{R}[x]$ belongs to $\mathbb{R}[x]$. Also, for every $a \in \mathbb{R}$, $ap$ is again a polynomial with coefficients in $\mathbb{R}$. 
Definition

Let $L$ be an $\mathbb{F}$-linear space. A subset $U$ of $L$ is a \textit{linear subspace} of $L$ (or just a subspace of $L$) if it satisfies the following conditions:

- if $x, y \in U$, then $x + y \in U$;
- if $a \in \mathbb{F}$ and $x \in U$, then $ax \in U$.

If $U$ is a subspace of a linear space $L$ and $x \in L$, we denote the set \{ $x + u \mid u \in U$ \} by $x + U$. 
Example

The set of polynomials $P_{\leq k}$ of degree less or equal to $k$ is a subspace of the linear space of polynomials. Indeed, $p, q \in P_{\leq k}$ their sum has degree less or equal to $k$; also, if $a \in \mathbb{R}$ and $p \in P_{\leq k}$, then $ap \in P_{\leq k}$.
The following statements are immediate for an $\mathbb{F}$-linear space $L$:

- the sets $L$ and $\{0_L\}$ are subspaces of $L$;
- each subspace $U$ of $L$ contains $0_L$. 
The subset \( \{0_L\} \) of any \( \mathbb{F} \)-linear space \( L \) is a subspace of \( L \) named the \textit{zero subspace}. This is the smallest subspace of \( L \).
**Theorem**

If \( \mathcal{L} = \{ L_i \mid i \in I \} \) is a collection of subspaces of an \( \mathbb{F} \)-linear space \( L \), then \( \bigcap \mathcal{L} \) is a subspace of \( L \).

**Proof.**

Suppose that \( \mathbf{x}, \mathbf{y} \in \bigcap \mathcal{L} \). Then, \( \mathbf{x}, \mathbf{y} \in L_i \), so \( \mathbf{x} + \mathbf{y} \in L_i \) and \( a\mathbf{x} \in L_i \) for every \( i \in I \). Thus, \( \mathbf{x} + \mathbf{y} \in \bigcap \mathcal{L} \) and \( a\mathbf{x} \in \bigcap \mathcal{L} \), which allows us to conclude that \( \bigcap \mathcal{L} \) is a subspace of \( L \).

Since \( L \) itself is a subspace of \( L \) it follows that the collection of subspaces of a linear space is a closure system \( \mathcal{C} \). If \( K_{\text{sub}} \) is the closure operator induced by \( \mathcal{C} \), then for every subset \( X \) of \( L \), \( K_{\text{sub}}(X) \) is the smallest subspace of \( L \) that contains \( X \).
Let $\text{SUBSP}(M)$ be the collection of subspaces of a linear space $M$. If this set is equipped with the inclusion relation $\subseteq$ (which is a partial order), then for any two subspaces $K, L$ both $\sup\{K, L\}$ and $\inf\{K, L\}$ exist and are given by:

\[
\begin{align*}
\sup\{K, L\} &= \{x + y \mid x \in K \text{ and } y \in L\} \\
\inf\{K, L\} &= K \cap L.
\end{align*}
\]
Let $H = \{ x + y \mid x \in K \text{ and } y \in L \}$. Observe that we have both $K \subseteq H$ and $L \subseteq H$ because $\mathbf{0}$ belongs to both $K$ and $L$.

If $u$ and $v$ belong to $H$, then $u = x_1 + y_1$ and $v = x_2 + y_2$, where $x_1, x_2 \in K$ and $y_1, y_2 \in L$. Since $x_1 + x_2 \in K$ and $y_1 + y_2 \in L$ (because $K$ and $L$ are subspaces), it follows that

$$u + v = x_1 + y_1 + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \in H.$$ 

We have $au = ax_1 + ax_2 \in H$ because $ax_1 \in K$ and $ax_2 \in L$. Thus, $H$ is a subspace of $M$ and is an upper bound of $\{K, L\}$ in the partially ordered set $(\text{SUBSP}(M), \subseteq)$.

If $G$ is a subspace of $M$ that contains both $K$ and $L$, then $x + y \in G$ for $x \in K$ and $y \in L$, so $H \subseteq G$. Thus, $H = \sup\{K, L\}$.

We denote $H = \sup\{K, L\}$ by $K + L$. 
Next, we prove the *modularity* of $\text{SUBSP}(M)$.

**Theorem**

Let $M$ be an $\mathbb{F}$-linear space. For any $P, Q, R \in \text{SUBSP}(M)$ such that $Q \subseteq P$ we have $P \cap (Q + R) = Q + (P \cap R)$.

**Proof.**

Note that $Q \subseteq P \cap (Q + R)$, $P \cap R \subseteq P \cap (Q + R)$. Therefore, we have the inclusion $Q + (P \cap R) \subseteq P \cap (Q + R) =$, which leaves us with the reverse inclusion to prove.

Let $z \in P \cap (Q + R)$. This implies $z \in P$ and $z = x + y$, where $x \in Q \subseteq P$ and $y \in R$. Therefore, $y = z - x \in P$, so $y \in P \cap R$. Consequently, $z \in Q + (P \cap R)$, so $P \cap (Q + R) \subseteq Q + (P \cap R)$. $\square$
Definition

If $L$ is an $\mathbb{F}$-linear space, and $X$ is a subset of $L$, an $X$-linear combination is an element $\mathbf{w}$ of $L$ that can be written as

$$\mathbf{w} = \sum_{i=1}^{n} c_i \mathbf{x}_i,$$

where $\mathbf{x}_i \in X$.

A linear combination of $L$ is an $X$-linear combination, where $X$ is a subset of $L$.

The set of all $X$-linear combinations is denoted by $\langle X \rangle$ and is referred to as the set spanned by $X$. 
Theorem

Let $L$ be an $\mathbb{F}$-linear space. If $X \subseteq L$, then $\langle X \rangle$ is the smallest subspace of $L$ that contains the set $X$. In other words, we have:

- $\langle X \rangle$ is a subspace of $L$;
- $X \subseteq \langle X \rangle$;
- if $X \subseteq M$, where $M$ is a subspace of $L$, then $\langle X \rangle \subseteq M$. 
Proof

It is clear that if $u$ and $v$ are two $X$-linear combinations, then $u + v$ and $au$ are also $X$-linear combinations, so $\langle X \rangle$ is a subspace of $L$.

For $x \in X$ we can write $1x = x$, so $X \subseteq \langle X \rangle$.

Finally, suppose that $X \subseteq M$, where $M$ is a subspace of $L$ and $a_1x_1 + \cdots + a_nx_n \in \langle X \rangle$, where $x_1, \ldots, x_n \in X$. Since $X \subseteq M$, we have $x_1, \ldots, x_n \in M$, hence $a_1x_1 + \cdots + a_nx_n \in M$ because $M$ is a subspace. Thus, $\langle X \rangle \subseteq M$. 
Let \( L \) be an \( \mathbb{F} \)-linear space. A finite subset \( U = \{x_1, \ldots, x_n\} \) of \( L \) is **linearly dependent** if \( a_1x_1 + \cdots + a_nx_n = 0_L \), where at least one element \( a_i \) of \( \mathbb{F} \) is not equal to 0.

If this condition is not satisfied then \( U \) is said to be **linearly independent**.

A set \( U \) that consists of one vector \( x \neq 0_L \) is linearly independent.
$U = \{x_1, \ldots, x_n\}$ of $L$ is linearly independent if $a_1x_1 + \cdots + a_nx_n = 0_L$ implies $a_1 = \cdots = a_n = 0$. Also, note that a set $U$ that is linearly independent does not contain $0_L$.

**Example**

Let $L$ be an $\mathbb{F}$-linear space. If $u \in L$, then the set $L_u = \{au \mid a \in \mathbb{F}\}$ is a linear subspace of $L$. Moreover, if $u \neq 0_L$, then the set $\{u\}$ is linearly independent. Indeed, if $au = 0_L$ and $a \neq 0$, then multiplying both sides of the above equality by $a^{-1}$ we obtain $(a^{-1}a)u = a^{-1}0$, or equivalently, $u = 0_L$, which contradicts the initial assumption. Thus, $\{u\}$ is a linearly independent set.
Definition

Let \( L \) be an \( \mathbb{F} \)-linear space. A subset \( W \) of \( L \) is \textit{linearly dependent} if it contains a finite subset \( U \) that is linearly dependent. A subset \( W \) is \textit{linearly independent} if it is not linearly dependent.

Thus, \( W \) is linearly independent if every finite subset of \( W \) is linearly independent. Further, any subset of a linearly independent subset is linearly independent and any superset of a linearly dependent set is linearly dependent.
Example

For every $\mathbb{F}$-linear space $L$ the set $\{0_L\}$ is linearly dependent because we have $10_L = 0_L$. 
Theorem

Let $L$ be an $\mathbb{F}$-linear space and let $W$ be a linearly independent subset of $L$. If $y$ is a linear combination

$$y = a_1x_1 + \cdots + a_nx_n,$$

for some finite subset $\{x_1, \ldots, x_n\}$ of $W$, then the coefficients $a_1, \ldots, a_n$ are uniquely determined.
Proof

Suppose that $y$ can be alternatively written as

$$y = b_1x_1 + \cdots + b_nx_n,$$

for some $b_1, \ldots, b_n \in \mathbb{F}$. Since $W$ is linearly independent this implies

$$(a_1 - b_1)x_1 + \cdots + (a_n - b_n)x_n = 0_L,$$

which, in turn, yields $a_1 - b_1 = \cdots = a_n - b_n = 0$. This, we have $a_i = b_i$ for $1 \leq i \leq n$. 
Definition

Let $\mathbb{F}$ be a field and let $L$ and $M$ be two $\mathbb{F}$-linear spaces. A **linear mapping** is a function $h : L \rightarrow M$ such that

$$h(ax + by) = ah(x) + bh(y)$$

for every scalars $a, b \in \mathbb{F}$ and $x, y \in L$.

An **affine mapping** is a function $f : L \rightarrow M$ such that there exists a linear mapping $h : L \rightarrow M$ and $b \in M$ such that $f(x) = h(x) + b$ for $x \in L$.

Linear mappings are also referred to as **linear spaces homomorphisms**, as **linear morphisms**, or as **linear operators**.

The set of morphisms between two $\mathbb{F}$-linear spaces $L$ and $M$ is denoted by $\text{Hom}(L, M)$. The set of affine mappings between two $\mathbb{F}$-linear spaces $L$ and $M$ is denoted by $\text{Aff}(L, M)$. 
Example

Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation defined by

$$h \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

This is a linear mapping $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. 
Define the mapping \( h : \mathbb{R}[x] \rightarrow \mathbb{R}[x] \) as

\[
h(p)(x) = \int_0^x p(t) \, dt.
\]

For example, for \( p(x) = x^2 + \frac{1}{3}x \) we have

\[
h(p)(x) = \int_0^x \left( t^2 + \frac{1}{3}t \right) \, dt = \frac{1}{3}x^3 + \frac{1}{6}x^2.
\]

It is easy to see that \( h(p_1 + p_2) = h(p_1) + h(p_2) \) and \( h(ap) = ah(p) \), which means that \( h \) is indeed a linear mapping.
The notion of subspace is closely linked to the notion of linear mapping as we show next.

**Theorem**

Let $L, M$ be two $\mathbb{F}$-linear spaces. If $h : L \rightarrow M$ is a linear mapping than the sets

$$Im(h) = \{ h(x) \mid x \in L \},$$

and

$$Ker(h) = \{ x \in L \mid h(x) = 0_M \}$$

are subspaces of the linear spaces $M$ and $L$, respectively.
Proof

Let \( u \) and \( v \) be two elements of \( \text{Im}(h) \). There exist \( x, y \in L \) such that \( u = h(x) \) and \( v = h(y) \). Since \( h \) is a linear mapping we have

\[
    u + v = h(x) + h(y) = h(x + y).
\]

Thus, \( u + v \in \text{Im}(h) \). Further, if \( a \in \mathbb{F} \), then \( au = ah(x) = h(ax) \), so \( au \in \text{Im}(h) \). Thus, \( \text{Im}(h) \) is indeed a subspace of \( M \).

Suppose now that \( s \) and \( t \) belong to \( \text{Ker}(h) \), that is \( h(s) = h(t) = 0_M \).

Then, \( h(s + t) = h(s) + h(t) = 0_M \), so \( s + t \in \text{Ker}(h) \). Also, \( h(as) = ah(s) = a0_M = 0_M \), which allows us to conclude that \( \text{Ker}(h) \) is a subspace of \( L \).
We refer to $\text{Im}(h)$ as the \textit{image} of $h$, and to $\text{Ker}(h)$ as the \textit{kernel} of $h$. 
Definition

Let \( h, g \in \text{Hom}(L, M) \) be two linear mappings between the \( \mathbb{F} \)-linear spaces \( L \) and \( M \). The \textit{sum} of \( h \) and \( g \) is the mapping \( h + g \) defined by
\[
(h + g)(x) = h(x) + g(x)
\]
for \( x \in L \).

If \( a \in \mathbb{F} \), the product \( af \) is defined as \( (af)(x) = af(x) \) for \( x \in L \).

If \( L, M \) are two \( \mathbb{F} \)-linear spaces, then the set \( \text{Hom}(L, M) \) is never empty because the zero morphism \( 0_{L,M} : L \to M \) defined as \( 0_{L,M}(x) = 0_M \) for \( x \in L \) is always an element of \( \text{Hom}(L, M) \).
Note that

\[(f + g)(ax + by) = f(ax + by) + g(ax + by)\]
\[= af(x) + bf(y) + ag(x) + bg(y)\]
\[= f(ax + by) + g(ax + by),\]

for all \(a, b \in F\) and \(x, y \in L\). This shows that the sum of two linear mappings is also a linear mapping.

**Theorem**

*Hom*(\(L, M\)) equipped with the sum and product defined above is an \(\mathbb{F}\)-linear space.

**Proof:** The zero element of *Hom*(\(L, M\)) is the mapping \(0_{L,M}\).
Definition

Let $L$ be an $F$-linear space. A linear form on $L$ is a morphism in $\text{Hom}(L, F)$, where the field $F$ is regarded as a linear space.
Definition

A *basis* of an \( \mathbb{F} \)-linear space \( L \) is a linearly independent subset \( W \) such that \( \langle W \rangle = L \).

If an \( \mathbb{F} \)-linear space \( L \) has a finite basis, then we say that \( L \) is a *linear space of finite type*.

Theorem

*Every non-zero \( \mathbb{F} \)-linear space \( L \) has a basis.*
Corollary

(Independent Set Extension Corollary) Let $L$ be an $\mathbb{F}$-linear space. If $W$ is a linearly independent set, then there exists a basis $T$ of $L$ such that $W \subseteq T$.

Proof: Since $W$ is a linearly independent set, if $\langle T \rangle = L$, then $W \cup T$ is also generating $L$. 
If an $\mathbb{F}$-linear space $L$ has a finite basis, then we say that $L$ is a linear space of finite type.

**Lemma**

Let $L$ be a finite type $\mathbb{F}$-linear space and let $T$ be a finite subset of $L$ that is not linearly independent. If $k = |T| \geq 2$ and $(t_1, \ldots, t_k)$ is a list of the vectors in $T$, then there exists a number $j$ such that $2 \leq j \leq m$ and $t_j$ is a linear combination of its predecessors in the sequence. Furthermore, we have $\langle T - \{t_j\} \rangle = \langle T \rangle$. 
Proof

Suppose that $T$ is linearly dependent. Then there exists a linear combination $\sum_{i=1}^{k} a^i t_i = 0_L$ such that some of the scalars $a^1, \ldots, a^k$ are different from 0. Let $j$ the largest number such that $1 \leq j \leq k$ and $a_j \neq 0$. The definition of $j$ implies that $a^1 t_1 + \cdots + a^j t_j = 0_L$, so

$$t_j = -\sum_{i=1}^{j-1} \frac{a^i}{a^j} t_i,$$

which shows that $t_j$ is a linear combination of its predecessors in the list. Consequently, the set of linear combinations of the vectors in $T - \{t_j\}$ equals $\langle T \rangle$. 
Theorem

(The Replacement Theorem) Let $L$ be a finite type $\mathbb{F}$-linear space such that the set $W$ spans the linear space $L$ and $|W| = n$. If $U$ is a linearly independent set in $V$ such that $|U| = m$, then $m \leq n$ and there exists a subset $W'$ of $W$ such that $W'$ contains $n - m$ vectors and $U \cup W'$ spans the space $L$. 
Proof

Suppose that $W = \{w_1, \ldots, w_n\}$ and $U = \{u_1, \ldots, u_m\}$. The argument is by induction on $m$. The basis case, $m = 0$, is immediate. Suppose the statement holds for $m$ and let $U = \{u_1, \ldots, u_m, u_{m+1}\}$ be a linearly independent set that contains $m + 1$ vectors. The set $\{u_1, \ldots, u_m\}$ is linearly independent, so by the inductive hypothesis $m \leq n$ and there exists a subset $W'$ of $W$ that contains $n - m$ vectors such that $\{u_1, \ldots, u_m\} \cup W'$ spans the space $L$. Without loss of generality we may assume that $W' = \{w_1, \ldots, w_{n-m}\}$. Thus, $u_{m+1}$ is a linear combination of the vectors of $\{u_1, \ldots, u_m, w_1, \ldots, w_{n-m}\}$, so we have

$$u_{m+1} = a^1 u_1 + \cdots + a^m u_m + b^1 w_1 + \cdots + b^{n-m} w_{n-m}.$$
Proof (cont’d)

We have $m + 1 \leq n$ because, otherwise, $m + 1 = n$ and $u_{m+1}$ would be a linear combination of $u_1, \ldots, u_m$, thereby contradicting the linear independence of the set $U$.

The set $\{u_1, \ldots, u_m, u_{m+1}, w_1, \ldots, w_{n-m}\}$ is not linearly independent. Let $v$ be the first member of the sequence $(u_1, \ldots, u_m, u_{m+1}, w_1, \ldots, w_{n-m})$ that is a linear combination of its predecessors. Then, $v$ cannot be one of the $u_i$ (with $1 \leq i \leq m$) because this would contradict the linear independence of the set $U$. Therefore, there exists $k$ such that $w_k$ is a linear combination of its predecessors and $1 \leq k \leq n - m$. By a previous lemma we can remove this element from the set $\{u_1, \ldots, u_m, u_{m+1}, w_1, \ldots, w_{n-m}\}$ without affecting the set spanned.
Corollary

Let \( L \) be a finite type \( \mathbb{F} \)-linear space and let \( U, W \) be two bases of \( L \). Then \(|U| = |W|\).

Proof.

Since \( U \) is a linearly independent set and \( \langle W \rangle = L \) we have \(|U| \leq |V|\). The reverse inequality, \(|V| \leq |U|\), is obtain by asserting that \( W \) is linearly independent and \( \langle U \rangle = L \). Thus, \(|U| = |W|\).

This allows the introduction of the notion of dimension for a linear space.

Definition

The *dimension* of a finite type linear space \( L \) is the number of elements of any basis of \( L \). The dimension of \( L \) is denoted by \( \dim(L) \).
If a linear space $L$ is not of finite type than we say that $\dim(L)$ is infinite.

**Theorem**

Let $L$ be an $\mathbb{F}$-linear space of finite type having the basis $B = \{x_1, \ldots, x_n\}$ and let $\{y_1, \ldots, y_n\}$ be a subset of an $\mathbb{F}$-linear space $M$. There exists a unique linear mapping $f : L \rightarrow M$ such that $f(x_i) = y_i$ for $1 \leq i \leq n$.

**Proof:** If $x \in L$ we have $x = a_1x_1 + \cdots + a_nx_n$ because $\{x_1, \ldots, x_n\}$ is a basis of $L$. Define $f(x)$ as $f(x) = \sum_{i=1}^{n} a_iy_i$. The uniqueness of the expression of $x$ as a linear combination of the elements of $B$ makes $f$ well-defined. The linearity of $f$ is immediate. For uniqueness, note that the value of $f$ is determined by the values of $f(x_i)$. 
Theorem

Let $L, M$ be two linear spaces of finite type with $\dim(L) = p$ and $\dim(M) = q$. Then, $\dim(\text{Hom}(L, M)) = pq$. 
Proof

Suppose that \( \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \) is a basis in \( L \) and \( \{\mathbf{y}_1, \ldots, \mathbf{y}_q\} \) is a basis in \( M \). For every \( i \) such that \( 1 \leq i \leq p \) and \( j \) such that \( 1 \leq j \leq q \) there exists a unique linear mapping \( f_{ij} : \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \rightarrow M \) such that:

\[
f_{ij}(\mathbf{x}_k) = \begin{cases} \mathbf{y}_j & \text{if } i = k, \\ 0_M & \text{otherwise,} \end{cases}
\]

for \( 1 \leq k \leq p \).

Note that if \( \mathbf{x} = \sum_{k=1}^{p} a_k \mathbf{x}_k \), the linearity of \( f_{ij} \) implies:

\[
f_{ij}(\mathbf{x}) = f_{ij} \left( \sum_{k=1}^{p} a_k \mathbf{x}_k \right) = \sum_{k=1}^{p} a_k f_{ij}(\mathbf{x}_k) = a_i f_{ij}(\mathbf{x}_i).
\]

We claim that the set \( \{f_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q\} \) is a basis for \( \text{Hom}(L, M) \).
Proof cont’d

Let \( f : L \rightarrow M \) be a linear mapping. If \( x \in L \) we can write
\[
x = \sum_{i=1}^{p} a_i x_i,
\]
so
\[
f(x) = \sum_{i=1}^{p} a_i f(x_i).
\]
In turn, since \( \{y_1, \ldots, y_q\} \) is a basis in \( M \),
\[
f(x_i) = \sum_{j=1}^{q} b_{ij} y_j,
\]
for some \( b_{ij} \in F \). This allows us to write:
\[
f(x) = \sum_{i=1}^{p} a_i \sum_{j=1}^{q} b_{ij} y_j = \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_{ij} y_j = \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_{ij} f_{ij}(x),
\]
which shows that each linear mapping in \( \text{Hom}(L, M) \) is a linear combination of functions \( f_{ij} \).
Furthermore, the set \( \{ f_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq p \} \) is linearly independent in \( \text{Hom}(L, M) \). Indeed, suppose that \( \sum_{i=1}^{p} \sum_{j=1}^{q} c_{ij} f_{ij}(x) = 0_M \). Then, for \( x = x_i \) we have \( \sum_{j=1}^{q} c_{ij} y_j = 0_M \), which implies \( c_{ij} = 0 \). We may conclude that \( \dim(\text{Hom}(L, M)) = \dim(L) \dim(M) \).
Theorem

If $W$ is a subspace of a finite type linear space $L$, then $\dim(W) \leq \dim(L)$.

Proof.

If $U$ is a linearly independent set in the subspace $W$, then it is clear that $U$ is linearly independent in $L$. There exists a basis $V$ of $L$ such that $U \subseteq V$ and $|V| = \dim(L)$. Therefore, $\dim(W) \leq \dim(L)$.
The notion of subspace is closely linked to the notion of linear mapping as we show next.

**Theorem**

Let $L, M$ be two $\mathbb{F}$-linear spaces. If $h : L \rightarrow M$ is a linear mapping then $\text{Im}(h)$ is a subspace of $M$ and $\text{Ker}(h)$ is a subspace of $L$. 
Proof

Let \( u \) and \( v \) be two elements of \( \text{Im}(h) \). There exist \( x, y \in L \) such that \( u = h(x) \) and \( v = h(y) \). Since \( h \) is a linear mapping we have

\[
u - v = h(x) - h(y) = h(x - y).
\]

Thus, \( u - v \in \text{Im}(h) \). Further, if \( a \in S \), then \( au = ah(x) = h(ax) \), so \( au \in \text{Im}(h) \). Thus, \( \text{Im}(h) \) is indeed a subspace of \( P \).

Suppose now that \( s \) and \( t \) belong to \( \text{Ker}(h) \), that is \( h(s) = h(t) = 0_M \). Then, \( h(s - t) = h(s) - h(t) = 0_M \), so \( s - t \in \text{Ker}(h) \). Also, \( h(as) = ah(s) = a0_M = 0_M \), which allows us to conclude that \( \text{Ker}(h) \) is a subspace of \( h \).
Theorem

Let $L$ and $M$ be two linear spaces, where $\dim(L) = n$, and let $h : L \longrightarrow M$ be a linear mapping. Then, we have

$$\dim(\ker(h)) + \dim(\text{Im}(h)) = n.$$
Proof

Suppose that \( \{ e_1, \ldots, e_m \} \) is a basis for the subspace \( \text{Ker}(h) \) of \( L \). Each such basis can be extended to a basis

\[
\{ e_1, \ldots, e_m, e_{m+1}, \ldots, e_n \}
\]

of the space \( L \). Any \( v \in L \) can be written as

\[
v = \sum_{i=1}^{n} a_i e_i.
\]

Since \( \{ e_1, \ldots, e_m \} \subseteq \text{Ker}(h) \) we have \( h(e_i) = 0_M \) for \( 1 \leq i \leq m \), so

\[
h(v) = \sum_{i=m+1}^{n} a_i h(e_i).
\]

This means that the set \( \{ h(e_{m+1}), \ldots, h(e_n) \} \) spans the subspace \( \text{Im}(h) \) of \( M \).
We show now that this set is linearly independent. Indeed, suppose that \( \sum_{i=m+1}^{n} b^i h(e_i) = 0 \). This implies \( h(\sum_{i=m+1}^{n} b^i e_i) = 0 \), that is, \( \sum_{i=m+1}^{n} b^i e_i \in \text{Ker}(h) \). Since \( \{e_1, \ldots, e_m\} \) is a basis for \( \text{Ker}(h) \) there exist \( m \) scalars \( c^1, \ldots, c^m \) such that

\[
\sum_{i=m+1}^{n} b^i e_i = c^1 e_1 + \cdots + c^m e_m.
\]

The fact that \( \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\} \) is a basis for \( L \) implies that \( c^1 = \cdots = c^m = b^{m+1} = \cdots = b^n = 0 \), so the set \( \{h(e_{m+1}), \ldots, h(e_n)\} \) is linearly independent and, therefore, a basis for \( \text{Im}(h) \). Thus, \( \dim(\text{Im}(h)) = n - m \), which concludes the argument.
Definition

Let $L$ and $M$ be two $\mathbb{F}$-linear spaces and let $h \in \text{Hom}(L, M)$. The rank of $h$ is $\text{rank}(h) = \dim(\text{Im}(h))$; the nullity of $h$ is $\text{nullity}(h) = \dim(\text{Ker}(h))$.

If $h : L \to M$ is a linear mapping and $L$ is a linear space of finite type, then

$$\dim(L) = \text{rank}(h) + \text{nullity}(h).$$
Theorem

Let \( h : L \rightarrow M \) be a linear mapping between two linear spaces. Then, \( \text{rank}(h) \leq \min\{\text{dim}(L), \text{dim}(M)\} \).

Proof.

It is clear that \( \text{rank}(h) \leq \text{dim}(L) \). On the other hand, \( \text{rank}(h) = \text{dim}(\text{Im}(h)) \leq \text{dim}(M) \) because \( \text{Im}(h) \) is a subspace of \( M \), so the inequality of the theorem follows.
Example

Let $L, M$ be two $\mathbb{F}$-linear spaces. For $h \in L^*$ and $y \in M$ define the mapping $\ell_{h,y}$ as $\ell_{h,y}(x) = h(x)y$ for $x \in L$. It is easy to verify that $\ell_{h,y}$ is a linear mapping, that is, $\ell_{h,y} \in \text{Hom}(L, M)$. Furthermore, we have $\text{rank}(\ell_{h,y}) = 1$ because $\text{Im}(\ell_{h,y})$ consists of the multiples of the vector $y$. 
Definition

Let $L$ and $M$ two $\mathbb{F}$-linear spaces. An *isomorphism* between these linear spaces is a linear mapping $h : L \rightarrow M$ that is a bijection. If an isomorphism exists between two $\mathbb{F}$-linear spaces $L$ and $M$ we say that these linear spaces are *isomorphic* and we write $L \cong M$.

Two $\mathbb{F}$-linear spaces that are isomorphic are indiscernible from an algebraic point of view.
If $L_1, L_2$ are subspaces of an $\mathbb{F}$-linear space $L$, then their intersection is non-empty because $0_L \in L_1 \cap L_2$. Moreover, it is easy to see that $L_1 \cap L_2$ is also a subspace of $L$.

Let $L_1, L_2$ be two subspaces of a linear space $L$. Their sum is the subset $L_1 + L_2$ of $L$ defined by

$$L_1 + L_2 = \{x + y \mid x \in L_1 \text{ and } y \in L_2\}.$$ 

It is immediate to verify that $L_1 + L_2$ is a subspace of $L$ and that $0_L \in L_1 \cap L_2$. 
Theorem

Let $L_1, L_2$ be two subspaces of the $\mathbb{F}$-linear space $L$. If $L_1 \cap L_2 = \{0_L\}$, then any vector $x \in L_1 + L_2$ can be uniquely written as $x = x_1 + x_2$, where $x_1 \in L_1$ and $x_2 \in L_2$.

Proof.

By the definition of the sum $L_1 + L_2$ it is clear that any vector $x \in L_1 + L_2$ can be written as $x = x_1 + x_2$. We need to prove only the uniqueness of $x_1$ and $x_2$.

Suppose that $x = x_1 + x_2 = y_1 + y_2$, where $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$. This implies $x_1 - y_1 = y_2 - x_2$ and, since $x_1 - y_1 \in L_1$ and $y_2 - x_2 \in L_2$, it follows that $x_1 - y_1 = y_2 - x_2 = 0_L$ by hypothesis. Therefore, $x_1 = y_1$ and $x_2 = y_2$.\qed
Theorem

Let $L_1, L_2$ be two subspaces of the $\mathbb{F}$-linear space $L$. If every vector $x \in L_1 + L_2$ can be uniquely written as $x = x_1 + x_2$, then $L_1 \cap L_2 = \{0\}$.

Proof.

Suppose that the uniqueness of the expression of $x$ holds but $z \in L_1 \cap L_2$ and $z \neq 0$. If $x = x_1 + x_2$, then we can also write $x = (x_1 + z) + (x_2 - z)$, where $x_1 + z \in L_1$ and $x_2 - z \in L_2$, $x_1 + z \neq x_1$ and $x_2 - z \neq x_2$, and this contradicts the uniqueness property. $\square$
Let $L$ be an $\mathbb{F}$-linear space. The set of linear forms defined on $L$ is denoted by $L^*$. This set has the natural structure of an $\mathbb{F}$-linear space known as the dual of the space $L$. The elements of $L^*$ are also referred to as covariant vectors or covectors. Frequently, we will refer to the vectors of the original linear space as contravariant vectors.
**Theorem**

Let \( B = \{ u_i \in L \mid 1 \leq i \leq n \} \) be a basis in an \( n \)-dimensional \( \mathbb{F} \)-linear space \( L \). If \( \{ a_i \in \mathbb{F} \mid 1 \leq i \leq n \} \) is a set of scalars, then there is a unique covector \( f \in L^* \) such that \( f(u_i) = a_i \) for \( 1 \leq i \leq n \).

**Proof.**

Since \( B \) is a basis in \( L \) we can write \( v = \sum_{i=1}^{n} c_i u_i \) for every \( v \in L \). Thus,

\[
f(v) = f \left( \sum_{i=1}^{n} c_i u_i \right) = \sum_{i=1}^{n} c_i a_i,
\]

which shows that the covector \( f \) is uniquely determined by the \( n \)-tuple of scalars \( a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \). \( \square \)
**Corollary**

Let $L$ be an $n$-dimensional $\mathbb{F}$-linear space. Then, its dual $L^*$ is isomorphic to $\mathbb{F}^n$, and, thus, $\dim(L^*) = \dim(L) = n$.

**Proof.**

The function $h : \mathbb{F}^n \rightarrow \text{Hom}(L, \mathbb{F})$ that maps the vector

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

to the function $f$ defined as

$$f(\mathbf{v}) = f \left( \sum_{i=1}^{n} c_i \mathbf{u}_i \right) = \sum_{i=1}^{n} c_i a_i,$$

where $B = \{ \mathbf{u}_i \in L \mid 1 \leq i \leq n \}$ is a basis in $L$ and $\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{u}_i$ is an isomorphism.
A linear form $f \in L^*$ is uniquely determined by its values on a basis of the space $L$. This allows us to prove the following extension theorem.

**Theorem**

Let $U$ be a subspace of a finite-dimensional $\mathbb{F}$-linear space $L$. A linear function $g : U \rightarrow \mathbb{F}$ belongs to $U^*$ if and only if there exists a linear form $f \in L^*$ such that $g$ is the restriction of $f$ to $U$. 
Proof

If \( g \) is the restriction of \( f \) to \( U \), then it is immediate that \( g \in U^* \).
Conversely, let \( g \in U^* \) and let \( B = \{u_1, \ldots, u_p\} \) be a basis of \( U \), where \( \dim(U) = p \). Consider an extension of \( B \) to a basis of the entire space \( B_1 = \{u_1, \ldots, u_p, u_{p+1}, \ldots, u_n\} \), where \( n = \dim(L) \) and define the linear form \( f : L \rightarrow \mathbb{F} \) by

\[ f(u_i) = \begin{cases} g(u_i) & \text{if } i \leq p, \\ 0 & \text{if } p + 1 \leq i \leq n. \end{cases} \]

Since \( f \) and \( g \) coincide for all members of the basis of \( U \) if follows that \( g \) is the restriction of \( f \) to \( U \).

We refer to \( f \) as the zero-extension of the linear form \( g \) defined on the subspace \( U \).
Theorem

If \( \{u_1, \ldots, u_n\} \) is a basis of the \( \mathbb{F} \)-linear space \( L \), then the set of linear forms \( \{f^j \mid 1 \leq j \leq n\} \) defined by

\[
f^j(u_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}
\]

is a basis of the dual linear space \( L^* \).
Proof

The set $F = \{f^1, \ldots, f^n\}$ spans the entire dual space $L^*$. Indeed, let $f \in L^*$ be defined by $f(u_i) = a_i$ for $1 \leq i \leq n$. Then, we have:

$$f(x) = a_1 f^1(x) + \cdots + a_n f^n(x)$$

for $x \in L$. Indeed, if $x = c^i u_i$, then

$$f(x) = f(c^i u_i) = c^i f(u_i) = c^i a_i.$$ 

On another hand,

$$a_i f^i(x) = a_i f^i(u_j) = a_i c^j f^i(u_j) = a_i c^i,$$

due to the definition of the linear forms $f_1, \ldots, f_n$. Therefore, $f = a_1 f^1 + \cdots + a_n f^n$, which shows that $\langle F \rangle = L^*$. 
Proof cont’d

To prove that the set $F$ is linearly independent in $L^*$ suppose that

$$a_1f^1 + \cdots + a_nf_n = 0_{L^*}.$$ 

This implies $a_1f^1(x) + \cdots + a_nf^n(x) = 0_L$ for every $x \in L$. Choosing $x = u_j$ we obtain $a_jf^j(u_j) = 0$, hence $a_j = 0$, and this can be shown for $1 \leq j \leq n$, which implies the linear independence.
The basis $F = \{f^1, \ldots, f^n\}$ of $L^*$ constructed before is the dual basis of the basis $U = \{u_1, \ldots, u_n\}$ of $L$. We refer to the pair $(U, F)$ as a pair of dual bases.

**Corollary**

The dual of an $n$-dimensional $\mathbb{F}$-linear space is an $n$-dimensional linear space.
Example

Let \( P_2[x] \) the linear space of polynomials of degree 2 in \( x \), that consists of polynomials of the form \( p(x) = ax^2 + bx + c \). The set \( \{p_0, p_1, p_2\} \) given by \( p_0(x) = 1, \ p_1(x) = x, \) and \( p_2(x) = x^2 \) is a basis in \( P_2[x] \). Note that we have

\[
\begin{align*}
  c &= p(0), \\
  b &= \frac{1}{2}(p(1) - p(-1)), \\
  a &= \frac{1}{2}(p(1) + p(-1) - 2p(0)).
\end{align*}
\]

If \( f : P_2[x] \rightarrow \mathbb{R} \) is a linear form we have

\[
  f(p) = af(x^2) + bf(x) + cf(0) \\
  = \frac{1}{2}(p(1) + p(-1) - 2p(0))f(x^2) + \frac{1}{2}(p(1) - p(-1))f(x) + p(0)f(1).
\]
Example cont’d

Example

Therefore, a basis in $P_2[x]^*$ consists of the functions

\[
\begin{align*}
    f^0(p) &= p(0), \\
    f^1(p) &= \frac{1}{2}(p(1) - p(-1)), \\
    f^2(p) &= \frac{1}{2}(p(1) + p(-1) - 2p(0)).
\end{align*}
\]
We saw that the dual $L^*$ of a $\mathbb{F}$-linear space $L$ is an $\mathbb{F}$-linear space. The construction of the dual may be repeated, and $L^{**}$, the dual of the dual $\mathbb{F}$-linear space is an $\mathbb{F}$-linear space. In the case of finite dimensional linear spaces we have $\dim(L^{**}) = \dim(L^*) = \dim(L)$, and all these spaces are isomorphic.

**Theorem**

Let $L$ be a finite-dimensional $\mathbb{F}$-linear space. Then, the dual $L^{**}$ of the dual $L^*$ of $L$ is an $\mathbb{F}$-linear space isomorphic to $L$. 
The notion of linear mapping can be extended as follows.

**Definition**

Let $L_1, \ldots, L_n, L$ be real linear spaces and let $L_1 \times \cdots \times L_n$ be the Cartesian product of the sets $L_1, \ldots, L_n$. An **real multilinear function** is a mapping $f : L_1 \times \cdots \times L_n \to L$ that is linear in each of its components when the other components are held fixed. In other words, $f$ satisfies the conditions:

$$f(x_1, \ldots, x_{i-1}, \sum_{j=1}^{k} a_j x_j^i, x_{i+1}, \ldots, x_n) = \sum_{j=1}^{k} a_j f(x_1, \ldots, x_{i-1}, x_j^i, x_{i+1}, \ldots, x_n),$$

for every $x_i, x_j^i \in L_i$ and $a_1, \ldots, a_k \in \mathbb{R}$. 
Definition

Let $L, M$ be two complex linear spaces. A function $f : L \times M \rightarrow \mathbb{C}$ is said to be Hermitian bilinear if it is linear in the first variable and skew-linear in the second, that is, it satisfies the equalities:

\[
\begin{align*}
  f(a_1x_1 + a_2x_2, y) &= a_1f(x_1, y) + a_2f(x_2, y), \\
  f(x, b_1y_1 + b_2y_2) &= \overline{b}_1f(x, y_1) + \overline{b}_2f(x, y_2)
\end{align*}
\]

for $a_1, a_2, b_1, b_2 \in \mathbb{C}$. 
The set of real multilinear functions defined on the linear spaces $L_1, \ldots, L_n$ and ranging in the real linear space $L$ is denoted by $\mathcal{M}(L_1, \ldots, L_n; L)$. The set of real multilinear forms is $\mathcal{M}(L_1, \ldots, L_n; \mathbb{R})$. 
Example

Multilinearity is distinct from the notion of linearity on a product of linear spaces. For instance, the mapping $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $h(x, y) = x + y$ is linear but not bilinear. On the other hand, the mapping $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $g(x, y) = xy$ is bilinear but not linear.
Definition

Let $L_1, \ldots, L_n, L$ be real linear spaces.
If $f, g \in \mathcal{M}(L_1, \ldots, L_n; L)$ are two multilinear functions, their sum is the function $f + g$ defined by

$$(f + g)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n),$$

and the product $af$, where $a \in \mathbb{F}$ is the function $af$ given by

$$(af)(x_1, \ldots, x_n) = af(x_1, \ldots, x_n)$$

for $x_i \in L_i$ and $1 \leq i \leq n$.

It is immediate to verify that $\mathcal{M}(L_1, \ldots, L_n; L)$ is an $\mathbb{R}$-linear space relative to these operations.
Let $f : L_1 \times L_2 \rightarrow L$ be a real bilinear function. Observe that for $x \in L_1$ and $y \in L_2$ we have:

\[
f(x, 0_{L_2}) = f(x, 0) = 0f(x, y) = 0_L \quad \text{and} \quad f(0_{L_1}, y) = f(0x, y) = 0f(x, y) = 0_L.
\]
Example

Let $L$ be an $\mathbb{R}$-linear space and let $\langle \cdot , \cdot \rangle : L^* \times L \rightarrow \mathbb{R}$ be the function given by $\langle h , y \rangle = h(y)$ for $h \in L^*$ and $y \in L$. It is immediate that $\langle \cdot , \cdot \rangle$ is a bilinear function because

\[
\langle ah + bg , y \rangle = a\langle h , y \rangle + b\langle g , y \rangle,
\]
\[
\langle h , ay + bz \rangle = a\langle h , y \rangle + b\langle h , z \rangle,
\]

for $a, b \in \mathbb{R}$, $h, g \in L^*$, and $y, z \in L$.

Moreover, we have $\langle h , y \rangle = 0$ for every $y \in L$ if and only if $h = 0_{L^*}$ and $\langle h , y \rangle = 0$ for every $h \in L^*$ if and only if $y = 0_L$. 
Example

Let $L_1, \ldots, L_n, L$ be $\mathbb{R}$-linear spaces, $a_i \in L_i$ for $1 \leq i \leq n$, and let $g_i \in L_i^*$. Define the function $G : L_1 \times L_n \longrightarrow \mathbb{R}$ as:

$$G(a_1, \ldots, a_n) = g_1(a_1) \cdots g_n(a_n)$$

for $a_i \in L_i$ and $1 \leq i \leq n$.

The function $G$ is multilinear. Indeed, if $a_i, b_i \in L_i$ and $a \in \mathbb{R}$ it is immediate to verify that

$$G(a_1, \ldots, a_i + b_i, \ldots, a_n) = G(a_1, \ldots, a_i, \ldots, a_n) + G(a_1, \ldots, b_i, \ldots, a_n),$$

and

$$G(a_1, \ldots, aa_i, \ldots, a_n) = aG(a_1, \ldots, a_i, \ldots, a_n).$$

Note, however, that $G$ is not a linear function because

$$G(aa_1, \ldots, aa_n) = a^n G(a_1, \ldots, a_n).$$
Example

The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x_1, x_2) = x_1x_2$ is bilinear because it is linear in each of its variables, separately, but is not linear in the ensemble of its arguments. Indeed, we have

$$f(x_1 + y_1, x_2) = f(x_1, x_2) + f(y_1, x_2),$$
$$f(x_1, x_2 + y_2) = f(x_1, x_2) + f(x_1, y_2)$$

for every $x_1, x_2, y_1, y_2 \in \mathbb{R}$, which shows the bilinearity of $f$. However, we have:

$$f(x_1 + x_2, y_1 + y_2) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$
$$\neq f(x_1, y_1) + f(x_2, y_2),$$

which means that $f$ is not a linear function.
Theorem

Let $U, V$ be two real linear spaces and let $\mathcal{M}(U, V; \mathbb{R})$ be the linear space of bilinear forms defined on $U \times V$. The linear spaces $\mathcal{M}(U, V; \mathbb{R})$, $	ext{Hom}(U, V^*)$ and $	ext{Hom}(V, U^*)$ are isomorphic.
Proof

It is immediate that $\Phi$ is a linear mapping because for $c, d \in \mathbb{R}$ and $h_1, h_2 \in \mathcal{M}(U, V; \mathbb{R})$ we have:

$$\Phi(ch_1 + dh_2)(a)(v) = ((ch_1 + dh_2)^a)(v)$$
$$= (ch_1 + dh_2)(a, v) = ch_1(a, v) + dh_2(a, v)$$
$$= ch_1^a(v) + dh_2^a(v)$$
$$= c\Phi(h_1)(a)(v) + d\Phi(h_2)(a)(v),$$

or

$$\Phi(ch_1 + dh_2) = c\Phi(h_1) + d\Phi(h_2).$$

Note that $\Phi$ maps $h : U \rightarrow V$ into the linear form that transforms $a$ into $h^a$ for $a \in U$. Thus, if $\Phi(h_1) = \Phi(h_2)$ we have both $h_1$ and $h_2$ yield equal values for $a \in U$, so $h_1 = h_2$, which proves the injectivity of $\Phi$. 
Let $f \in \text{Hom}(U, V^*)$. For every $a \in U$ there exists a linear form $g : V \rightarrow \mathbb{R}$ such that $f(a) = g$, or $f(a)(v) = g(v)$ for every $v \in V$. The mapping $h : U \times V \rightarrow \mathbb{R}$ defined by $h(u, bfv) = f(u)(v)$ is bilinear and $\Phi(h)(u)(v) = h^u(v) = h(u, v) = f(u)(v)$, which means that $\Phi(h) = f$. Thus, $\Phi$ is also surjective and, therefore, it is an isomorphism between the linear spaces $\mathfrak{M}(U, V; \mathbb{R})$, and $\text{Hom}(U, V^*)$. The existence of an isomorphism between and $\text{Hom}(V, U^*)$ has a similar argument.
The linear space $\mathcal{M}(U, V; \mathbb{R})$ will also be denoted by $U^* \otimes V^*$. We will refer to this space as the tensor product of the spaces $U$ and $V$.

**Corollary**

Let $U, V$ be two $\mathbb{R}$-linear spaces. Then, $\dim(U \otimes V) = \dim(U) \cdot \dim(V)$.

**Proof.**

Since $\dim(V^*) = \dim(V) = n$, we have $\dim(\text{Hom}(U, V^*)) = mn$. The result follows immediately.
Let $U, V, W$ be three $\mathbb{R}$-linear spaces of finite dimensions having the bases 
\{\textbf{u}_1, \ldots, \textbf{u}_m\}, \{\textbf{v}_1, \ldots, \textbf{v}_n\} and \{\textbf{w}_1, \ldots, \textbf{w}_p\}$, respectively, and let \(f : U \times V \to W\) be a bilinear function. If 
\[u = \sum_{i=1}^{m} a_i \textbf{u}_i \in U, \quad v = \sum_{j=1}^{n} b_j \textbf{v}_j,\]
then 
\[f(u, v) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j f(\textbf{u}_i, \textbf{v}_j).\]

Since \(f(\textbf{u}_i, \textbf{v}_j) \in W\) there exist \(c_{ij}^k\) such that 
\[f(\textbf{u}_i, \textbf{v}_j) = \sum_{k=1}^{p} c_{ij}^k \textbf{w}_k,\]
hence 
\[f(u, v) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} a_i b_j c_{ij}^k \textbf{w}_k.\]

Thus, the set \(\{c_{ij}^k \in \mathbb{R} \mid 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\}\) (which contains \(mnp\) elements) determines a bilinear function relative to the chosen bases in $U, V$ and $W$. 
Unlike the case $n = 1$, the set of values of a multilinear function $f : M_1 \times \cdots \times M_n \longrightarrow M$ is not a subspace of $M$ in general. Indeed, consider a two-dimensional $\mathbb{R}$-linear space $U$ having a basis $\{u_1, u_2\}$, a four-dimensional $\mathbb{R}$-linear space $W$ having the basis $\{w_1, w_2, w_3, w_4\}$, and the bilinear function $f : U \times U \longrightarrow W$ defined as:

$$f(u, v) = u_1 v_1 w_1 + u_1 v_2 w_2 + u_2 v_1 w_3 + u_2 v_2 w_4,$$

where $u = u_1 u_1 + u_2 u_2$ and $v = v_1 u_1 + v_2 u_2$. 
Let $S$ be the set of all vectors of the form $s = f(u, v)$. By the definition of $S$ there exist $u, v \in U$ such that

\[ s_1 = u_1 v_1, \ s_2 = u_1 v_2, \ s_3 = u_2 v_1, \ s_4 = u_2 v_2, \]

hence $s_1 s_4 = s_2 s_3$ for any $s \in S$.

Define the vectors $z, t$ in $W$ as

\[ z = 2w_1 + 2w_2 + w_3 + w_4, \]
\[ t = w_1 + w_3. \]

Note that we have both $z \in S$ and $t \in S$. However,

\[ x = z - t = w_1 + 2w_2 + w_4 \]

does not belong to $S$ because $x_1 x_4 = 1$ and $x_2 x_3 = 0$. 
Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bilinear form. Since the vectors
\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
form a basis in $\mathbb{R}^2$, $f$ can be written as
\[
f(ae_1 + be_2, ce_1 + de_2) = af(e_1, ce_1 + de_2) + bf(e_2, ce_1 + de_2)
\]  
\[
= acf(e_1, e_1) + adf(e_1, e_2) + bcf(e_2, e_1) + bdf(e_2, e_2)
\]  
\[
= \alpha f(e_1, e_1) + \beta f(e_1, e_2) + \gamma f(e_2, e_1) + \delta f(e_2, e_2),
\]
where
\[
\alpha = ac, \beta = ad, \gamma = bc, \delta = bd.
\]
Thus, the multilinearity of $f$ implies $\alpha \delta = \beta \gamma$. 