CS724: Topics in Algorithms
Norms and Inner Products - I
Slide Set 4

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1. Basic Inequalities

2. Metric Spaces

3. Norms
Lemma

Let $p, q \in \mathbb{R} - \{0, 1\}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have $p > 1$ if and only if $q > 1$. Furthermore, one of the numbers $p, q$ belongs to the interval $(0, 1)$ if and only if the other number is negative.
Lemma

Let $p, q \in \mathbb{R} - \{0, 1\}$ be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Then, for every $a, b \in \mathbb{R}_{\geq 0}$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where the equality holds if and only if $a = b^{-\frac{1}{1-p}}$. 
Proof

We have $q > 1$. Consider the function $f(x) = \frac{x^p}{p} + \frac{1}{q} - x$ for $x \geq 0$. We have $f'(x) = x^{p-1} - 1$, so the minimum is achieved when $x = 1$ and $f(1) = 0$. Thus,

$$f \left( ab^{\frac{1}{p-1}} \right) \geq f(1) = 0,$$

which amounts to

$$\frac{a^p b^{\frac{p}{p-1}}}{p} + \frac{1}{q} - ab^{\frac{1}{p-1}} \geq 0.$$

By multiplying both sides of this inequality by $b^{\frac{p}{p-1}}$, we obtain the desired inequality.
Observe that if \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p < 1 \), then \( q < 0 \). In this case, we have the reverse inequality

\[
ab \geq \frac{a^p}{p} + \frac{b^q}{q}.
\] (1)

which can be shown by observing that the function \( f \) has a maximum in \( x = 1 \). The same inequality holds when \( q < 1 \) and therefore \( p < 0 \).
**Theorem**

*(The Hölder Inequality)* Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ nonnegative numbers, and let $p$ and $q$ be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. We have

$$
\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.
$$
Proof

If \( a_1 = \cdots = a_n = 0 \) or if \( b_1 = \cdots = b_n = 0 \), then the inequality is clearly satisfied. Therefore, we may assume that at least one of \( a_1, \ldots, a_n \) and at least one of \( b_1, \ldots, b_n \) is non-zero. Define the numbers

\[
x_i = \frac{a_i}{\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}}} \quad \text{and} \quad y_i = \frac{b_i}{\left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}}
\]

for \( 1 \leq i \leq n \). Lemma on Slide 3 applied to \( x_i, y_i \) yields

\[
\frac{a_i b_i}{\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{a_i^p}{\sum_{i=1}^n a_i^p} + \frac{1}{q} \frac{b_i^p}{\sum_{i=1}^n b_i^p}.
\]

Adding these inequalities, we obtain

\[
\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}
\]

because \( \frac{1}{p} + \frac{1}{q} = 1 \).
The nonnegativity of the numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ can be relaxed by using absolute values.

**Theorem**

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ numbers and let $p$ and $q$ be two numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. We have

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}}.$$
Proof

By a previous theorem, we have:

\[
\sum_{i=1}^{n} |a_i||b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}.
\]

The needed equality follows from the fact that

\[
\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sum_{i=1}^{n} |a_i||b_i|.
\]
Corollary

(The Cauchy-Schwarz Inequality for $\mathbb{R}^n$) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ real numbers. We have

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}.$$ 

Proof.

The inequality follows immediately by taking $p = q = 2$. □
Theorem

(Minkowski’s Inequality) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ nonnegative real numbers. If $p \geq 1$, we have

$$\left( \sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}}.$$ 

If $p < 1$, the inequality sign is reversed.
Proof

For $p = 1$, the inequality is immediate. Therefore, we can assume that $p > 1$. Note that

$$
\sum_{i=1}^{n} (a_i + b_i)^p = \sum_{i=1}^{n} a_i (a_i + b_i)^{p-1} + \sum_{i=1}^{n} b_i (a_i + b_i)^{p-1}.
$$

By Hölder’s inequality for $p, q$ such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$
\sum_{i=1}^{n} a_i (a_i + b_i)^{p-1} \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} (a_i + b_i)^{(p-1)q} \right)^{\frac{1}{q}}
$$

$$
= \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{q}}.
$$
Proof cont’d

Similarly, we can write

\[ \sum_{i=1}^{n} b_i(a_i + b_i)^{p-1} \leq \left( \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{q}}. \]

Adding the last two inequalities yields

\[ \sum_{i=1}^{n} (a_i + b_i)^p \leq \left( \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}} \right) \left( \sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{q}}, \]

which is equivalent to inequality

\[ \left( \sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}}. \]
Definition

A function $d : S^2 \rightarrow \mathbb{R}_{\geq 0}$ is a *metric* if it has the following properties:

- $d(x, y) = 0$ if and only if $x = y$ for $x, y \in S$;
- $d(x, y) = d(y, x)$ for $x, y \in S$;
- $d(x, y) \leq d(x, z) + d(z, y)$ for $x, y, z \in S$.

The pair $(S, d)$ will be referred to as a *metric space*. 
If property (i) is replaced by the weaker requirement that \( d(x, x) = 0 \) for \( x \in S \), then we refer to \( d \) as a *semimetric* on \( S \). Thus, if \( d \) is a semimetric \( d(x, y) = 0 \) does not necessarily imply \( x = y \) and we can have for two distinct elements \( x, y \) of \( S \), \( d(x, y) = 0 \). If \( d \) is a semimetric, then we refer to the pair \((S, d)\) as a *semimetric space*. 
Example

Let $S$ be a nonempty set. Define the mapping $d : S^2 \rightarrow \mathbb{R}_{\geq 0}$ by

$$d(u, v) = \begin{cases} 1 & \text{if } u \neq v, \\ 0 & \text{otherwise,} \end{cases}$$

for $x, y \in S$. It is easy to see that $d$ satisfies the definiteness property. To prove that $d$ satisfies the triangular inequality, we need to show that

$$d(x, y) \leq d(x, z) + d(z, y)$$

for all $x, y, z \in S$. This is clearly the case if $x = y$. Suppose that $x \neq y$, so $d(x, y) = 1$. Then, for every $z \in S$, we have at least one of the inequalities $x \neq z$ or $z \neq y$, so at least one of the numbers $d(x, z)$ or $d(z, y)$ equals 1. Thus $d$ satisfies the triangular inequality. The metric $d$ introduced here is the \textit{discrete metric} on $S$. 
Example

Consider the mapping \( d : (\text{Seq}_n(S))^2 \rightarrow \mathbb{R}_{\geq 0} \) defined by

\[
d(p, q) = |\{i \mid 0 \leq i \leq n - 1 \text{ and } p(i) \neq q(i)\}|
\]

for all sequences \( p, q \) of length \( n \) on the set \( S \).

It is easy to see that \( d \) is a metric. We justify here only the triangular inequality. Let \( p, q, r \) be three sequences of length \( n \) on the set \( S \). If \( p(i) \neq q(i) \), then \( r(i) \) must be distinct from at least one of \( p(i) \) and \( q(i) \). Therefore,

\[
\{i \mid 0 \leq i \leq n - 1 \text{ and } p(i) \neq q(i)\} \\
\subseteq \{i \mid 0 \leq i \leq n - 1 \text{ and } p(i) \neq r(i)\} \cup \{i \mid 0 \leq i \leq n - 1 \text{ and } r(i) \neq q(i)\}
\]

which implies the triangular inequality.
Example

For \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) the *Euclidean metric* is the mapping

\[
    d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.
\]

The first two conditions of Definition 7 are obviously satisfied. To prove the third inequality, let \( x, y, z \in \mathbb{R}^n \). Choosing \( a_i = x_i - y_i \) and \( b_i = y_i - z_i \) for \( 1 \leq i \leq n \) in Minkowski’s inequality implies

\[
    \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2},
\]

which amounts to \( d(x, z) \leq d(x, y) + d(y, z) \). Thus, we conclude that \( d \) is indeed a metric on \( \mathbb{R}^n \).
We use frequently use the notions of closed sphere and open sphere.

**Definition**

Let \((S, d)\) be a metric space. The **closed sphere** centered in \(x \in S\) of radius \(r\) is the set

\[
B_d[x, r] = \{y \in S | d(x, y) \leq r\}.
\]

The **open sphere** centered in \(x \in S\) of radius \(r\) is the set

\[
B_d(x, r) = \{y \in S | d(x, y) < r\}.
\]
Definition

Let \((S, d)\) be a metric space. The *diameter* of a subset \(U\) of \(S\) is the number \(\text{diam}_{S,d}(U) = \sup\{d(x, y) \mid x, y \in U\}\). The set \(U\) is *bounded* if \(\text{diam}_{S,d}(U)\) is finite.

The *diameter* of the metric space \((S, d)\) is the number

\[
\text{diam}_{S,d} = \sup\{d(x, y) \mid x, y \in S\}.
\]

If the metric space is clear from the context, then we denote the diameter of a subset \(U\) just by \(\text{diam}(U)\).

If \((S, d)\) is a finite metric space, then \(\text{diam}_{S,d} = \max\{d(x, y) \mid x, y \in S\}\).
A mapping \( d : S \times S \rightarrow \mathbb{R}_{\geq 0} \) can be extended to the set of subsets of \( S \) by defining \( d(U, V) \) as

\[
d(U, V) = \inf \{ d(u, v) \mid u \in U \text{ and } v \in V \}
\]

for \( U, V \in \mathcal{P}(S) \).

Observe that, even if \( d \) is a metric, then its extension is not, in general, a metric on \( \mathcal{P}(S) \) because it does not satisfy the triangular inequality. Instead, we can show that for every \( U, V, W \) we have

\[
d(U, W) \leq d(U, V) + \text{diam}(V) + d(V, W).
\]
Indeed, by the definition of $d(U, V)$ and $d(V, W)$, for every $\epsilon > 0$, there exist $u \in U$, $v, v' \in V$, and $w \in W$ such that

\[
\begin{align*}
\quad\quad\quad\quad\quad\quad d(U, V) & \leq d(u, v) \leq d(U, V) + \frac{\epsilon}{2}, \\
\quad d(V, W) & \leq d(v', w) \leq d(V, W) + \frac{\epsilon}{2}.
\end{align*}
\]

By the triangular axiom, we have

\[
d(u, w) \leq d(u, v) + d(v, v') + d(v', w).
\]

Hence,

\[
d(u, w) \leq d(U, V) + diam(V) + d(V, W) + \epsilon,
\]

which implies

\[
d(U, W) \leq d(U, V) + diam(V) + d(V, W) + \epsilon
\]

for every $\epsilon > 0$. This yields the needed inequality.
Definition

Let $(S, d)$ be a metric space. The sets $U, V \in \mathcal{P}(S)$ are separate if $d(U, V) > 0$.

We denote the number $d(\{u\}, V) = \inf\{d(u, v) \mid v \in V\}$ by $d(u, V)$. It is clear that $u \in V$ implies $d(u, V) = 0$. 
The notion of dissimilarity is a generalization of the notion of metric.

**Definition**

A **dissimilarity on a set** $S$ is a function $d : S^2 \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following conditions:

- $d(x, x) = 0$ for all $x \in S$;
- $d(x, y) = d(y, x)$ for all $x, y \in S$.

The pair $(S, d)$ is a **dissimilarity space**.
A related concept is the notion of similarity.

**Definition**

A *similarity on a set* $S$ is a function $s : S^2 \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following conditions:

- $s(x, y) \leq s(x, x) = 1$ for all $x, y \in S$;
- $s(x, y) = s(y, x)$ for all $x, y \in S$.

The pair $(S, s)$ is a *similarity space*. 
Example

Let \( d : S^2 \rightarrow \mathbb{R}_{\geq 0} \) be a metric on the set \( S \). Then \( s : S^2 \rightarrow \mathbb{R}_{\geq 0} \) defined by \( s(x, y) = 2^{-d(x,y)} \) for \( x, y \in S \) is a dissimilarity, such that \( s(x, x) = 1 \) for every \( x, y \in S \).
Definition

A *seminorm* on an $F$-linear space $V$ is a mapping $\nu : V \rightarrow \mathbb{R}$ that satisfies the following conditions:

- $\nu(x + y) \leq \nu(x) + \nu(y)$ (subadditivity), and
- $\nu(ax) = |a|\nu(x)$ (positive homogeneity),

for $x, y \in V$ and $a \in F$.

By taking $a = 0$ in the second condition of the definition we have $\nu(0) = 0$ for every seminorm on a real or complex space.
Theorem

If $V$ is a real or complex linear space and $\nu : V \to \mathbb{R}$ is a seminorm on $V$, then

$$\nu(x - y) \geq |\nu(x) - \nu(y)|,$$

for $x, y \in V$.

Proof.

We have $\nu(x) \leq \nu(x - y) + \nu(y)$, so

$$\nu(x) - \nu(y) \leq \nu(x - y). \quad (2)$$

Since $\nu(x - y) = |-1|\nu(y - x) \geq \nu(y) - \nu(x)$ we have

$$-(\nu(x) - \nu(y)) \leq \nu(x) - \nu(y). \quad (3)$$

The Inequalities (2) and (3) give the desired inequality.
Corollary

If \( p : V \rightarrow \mathbb{R} \) is a seminorm on \( V \), then \( p(x) \geq 0 \) for \( x \in V \).

Proof.
By choosing \( y = 0 \) we have \( \nu(x) \geq |\nu(x)| \geq 0 \).
Definition

Let $\mathcal{F} = (F, \{0, 1, +, -, \cdot\})$ be the real or the complex field. A norm on an $F$-linear space $V$ is a seminorm $\nu : V \to \mathbb{R}$ such that $\nu(x) = 0$ implies $x = 0$ for $x \in V$.

The pair $(V, \nu)$ is referred to as a normed linear space.
Example

The set of real-valued continuous functions defined on the interval \([-1, 1]\) is a real linear space. The addition of two such functions \(f, g\), is defined by 
\[(f + g)(x) = f(x) + g(x)\] for \(x \in [-1, 1]\); the multiplication of \(f\) by a scalar \(a \in \mathbb{R}\) is 
\[(af)(x) = af(x)\] for \(x \in [-1, 1]\).

Define \(\nu(f) = \sup\{|f(x)| \mid x \in [-1, 1]\}\). Since \(|f(x)| \leq \nu(f)\) and 
\(|g(x)| \leq \nu(g)\) for \(x \in [-1, 1]\), it follows that 
\[|(f + g)(x)| \leq |f(x)| + |g(x)| \leq \nu(f) + \nu(g)\]. Thus, 
\[\nu(f + g) \leq \nu(f) + \nu(g)\].

We denote \(\nu(f)\) by \(\|f\|\).
Theorem

For $p \geq 1$, the function $\nu_p : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$
\nu_p(x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}},
$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, is a norm on $\mathbb{R}^n$. 

Proof

We must prove that \( \nu_p \) satisfies the conditions of the definition of norms and that \( \nu_p(x) = 0 \) implies \( x = 0 \).

Let \( x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \in \mathbb{R}^n \). Minkowski’s inequality applied to the nonnegative numbers \( a_i = |x_i| \) and \( b_i = |y_i| \) amounts to

\[
\left( \sum_{i=1}^{n} (|x_i| + |y_i|)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}.
\]

Since \( |x_i + y_i| \leq |x_i| + |y_i| \) for every \( i \), we have

\[
\left( \sum_{i=1}^{n} (|x_i + y_i|)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}},
\]

that is, \( \nu_p(x + y) \leq \nu_p(x) + \nu_p(y) \). Thus, \( \nu_p \) is a norm on \( \mathbb{R}^n \).
Example

The mapping $\nu_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\nu_1(x) = |x_1| + |x_2| + \cdots + |x_n|,$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. $\nu_1$ is a norm on $\mathbb{R}^n$. 
**Example**

A special norm on $\mathbb{R}^n$ is the function $\nu_\infty : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ given by

$$\nu_\infty(x) = \max\{|x_i| \mid 1 \leq i \leq n\}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We start from the inequality

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \nu_\infty(x) + \nu_\infty(y)$$

for every $i$, $1 \leq i \leq n$. This implies

$$\nu_\infty(x + y) = \max\{|x_i + y_i| \mid 1 \leq i \leq n\} \leq \nu_\infty(x) + \nu_\infty(y),$$

which gives the desired inequality.
Example

This norm can be regarded as a limit case of the norms $\nu_p$. Indeed, let $\mathbf{x} \in \mathbb{R}^n$ and let $M = \max \{|x_i| \mid 1 \leq i \leq n\} = |x_{\ell_1}| = \cdots = |x_{\ell_k}|$ for some $\ell_1, \ldots, \ell_k$, where $1 \leq \ell_1, \ldots, \ell_k \leq n$. Here $x_{\ell_1}, \ldots, x_{\ell_k}$ are the components of $\mathbf{x}$ that have the maximal absolute value and $k \geq 1$. We can write

$$\lim_{p \to \infty} \nu_p(\mathbf{x}) = \lim_{p \to \infty} M \left( \sum_{i=1}^{n} \left( \frac{|x_i|}{M} \right)^p \right)^{\frac{1}{p}} = \lim_{p \to \infty} M(k)^{\frac{1}{p}} = M,$$

which justifies the notation $\nu_\infty$. 
We use the alternative notation $\| x \|_p$ for $\nu_p(x)$. We refer $\| x \|_2$ as the \textit{Euclidean norm} of $x$ and we denote this norm simply by $\| x \|$ when there is no risk of confusion.
Example

For $p \geq 1$, let $\ell_p$ be the set that consists of sequences of real numbers $x = (x_0, x_1, \ldots)$ such that the series $\sum_{i=0}^{\infty} |x_i|^p$ is convergent. We can show that $\ell_p$ is a linear space.

Let $x, y \in \ell_p$ be two sequences in $\ell_p$. Using Minkowski’s inequality we have

$$
\sum_{i=0}^{n} |x_i + y_i|^p \leq \sum_{i=0}^{n} (|x_i| + |y_i|)^p \leq \sum_{i=0}^{n} |x_i|^p + \sum_{i=0}^{n} |y_i|^p,
$$

which shows that $x + y \in \ell_p$. It is immediate that $x \in \ell_p$ implies $ax \in \ell_p$ for every $a \in \mathbb{R}$ and $x \in \ell_p$. 
The following statement shows that any norm defined on a linear space generates a metric on the space.

**Theorem**

Each norm $\nu : V \rightarrow \mathbb{R}_{\geq 0}$ on a real linear space $V$ generates a metric on the set $V$ defined by $d_\nu(x, y) = \nu(x - y)$ for $x, y \in V$.

**Proof.**

Note that if $d_\nu(x, y) = \nu(x - y) = 0$, it follows that $x - y = 0$; that is, $x = y$.

The symmetry of $d_\nu$ is obvious and so we need to verify only the triangular axiom. Let $x, y, z \in L$. Applying the subaditivity of norms we have

$$
\nu(x - z) = \nu(x - y + y - z) \leq \nu(x - y) + \nu(y - z)
$$

or, equivalently, $d_\nu(x, z) \leq d_\nu(x, y) + d_\nu(y, z)$, for every $x, y, z \in L$, which concludes the argument.
Observe that the norm $\nu$ can be expressed using $d_\nu$ as

$$\nu(x) = d_\nu(x, 0)$$

for $x \in V$.

For $p \geq 1$, then $d_p$ denotes the metric $d_{\nu_p}$ induced by the norm $\nu_p$ on the linear space $\mathbb{R}^n$ known as the Minkowski metric.

If $p = 2$, we have the Euclidean metric on $\mathbb{R}^n$ given by

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$
For $p = 1$, we have

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$ 

This metric is known also as the \textit{city-block metric}.

The norm $\nu_\infty$ generates the metric $d_\infty$ given by

$$d_\infty(x, y) = \max\{|x_i - y_i| \mid 1 \leq i \leq n\},$$

also known as the \textit{Chebyshev metric}.
A representation of these metrics can be seen below for the special case of $\mathbb{R}^2$. If $\mathbf{x} = (x_0, x_1)$ and $\mathbf{y} = (y_0, y_1)$, then $d_2(\mathbf{x}, \mathbf{y})$ is the length of the hypotenuse of the right triangle and $d_1(\mathbf{x}, \mathbf{y})$ is the sum of the lengths of the two legs of the triangle.
Theorem

(Projections on Closed Sets Theorem) Let $U$ be a closed subset of $\mathbb{R}^n$ such that $U \neq \emptyset$ and let $x_0 \in \mathbb{R}^n - U$. Then, there exists $x_1 \in U$ such that $\| x - x_0 \|_2 \geq \| x_1 - x_0 \|_2$ for every $x \in U$. 
Proof

Let \( d = \inf \left\{ \| x - x_0 \|_2 \mid x \in U \right\} \) and let \( U_n = U \cap B \left( x_0, d + \frac{1}{n} \right) \). Note that the sets form a descending sequence of bounded and closed sets \( U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq \cdots \). Since \( U_1 \) is compact, \( \bigcap_{n \geq 1} U_n \neq \emptyset \). Let \( x_1 \in \bigcap_{n \geq 1} U_n \). Since \( U_n \subseteq U \) for every \( n \), it follows that \( x_1 \in U \).

Note that \( \| x_1 - x_0 \|_2 \leq d + \frac{1}{n} \) for every \( n \) because \( x_1 \in U_n = U \cap B \left( x_0, d + \frac{1}{n} \right) \). This implies \( \| x_1 - x_0 \|_2 \leq d \leq \| x - x_0 \|_2 \) for every \( x \in U \).
Lemma

Let $a_1, \ldots, a_n$ be $n$ positive numbers. If $p$ and $q$ are two positive numbers such that $p \leq q$, then

$$(a_1^p + \cdots + a_n^p)^{\frac{1}{p}} \geq (a_1^q + \cdots + a_n^q)^{\frac{1}{q}}.$$

Proof: Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be the function defined by

$$f(r) = (a_1^r + \cdots + a_n^r)^{\frac{1}{r}}.$$

Since

$$\ln f(r) = \frac{\ln (a_1^r + \cdots + a_n^r)}{r},$$

it follows that

$$\frac{f'(r)}{f(r)} = -\frac{1}{r^2} \ln (a_1^r + \cdots + a_n^r) + \frac{1}{r} \cdot \frac{a_1^r \ln a_1 + \cdots + a_n^r \ln a_n}{a_1^r + \cdots + a_n^r}.$$
Proof cont’d

To prove that \( f'(r) < 0 \), it suffices to show that

\[
\frac{a_1^r \ln a_1 + \cdots + a_n^r \ln a_n}{a_1^r + \cdots + a_n^r} \leq \frac{\ln (a_1^r + \cdots + a_n^r)}{r}.
\]

This last inequality is easily seen to be equivalent to

\[
\sum_{i=1}^{n} \frac{a_i^r}{a_1^r + \cdots + a_n^r} \ln \frac{a_i^r}{a_1^r + \cdots + a_n^r} \leq 0,
\]

which holds because

\[
\frac{a_i^r}{a_1^r + \cdots + a_n^r} \leq 1
\]

for \( 1 \leq i \leq n \).
Theorem

Let $p$ and $q$ be two positive numbers such that $p \leq q$. For every $\mathbf{u} \in \mathbb{R}^n$, we have $\| \mathbf{u} \|_p \geq \| \mathbf{u} \|_q$.

Proof.
This statement follows immediately from previous Lemma.
Corollary

Let $p, q$ be two positive numbers such that $p \leq q$. For every $x, y \in \mathbb{R}^n$, we have $d_p(x, y) \geq d_q(x, y)$.

Proof.

This statement follows immediately from the previous Theorem.
Example

For $p = 1$ and $q = 2$ the inequality of the Theorem becomes

$$\sum_{i=1}^{n} |u_i| \leq \sqrt{\sum_{i=1}^{n} |u_i|^2},$$

which is equivalent to

$$\frac{\sum_{i=1}^{n} |u_i|}{n} \leq \sqrt{\frac{\sum_{i=1}^{n} |u_i|^2}{n}}.$$
Theorem

Let $p \geq 1$. For every $x \in \mathbb{R}^n$ we have

$$\| x \|_\infty \leq \| x \|_p \leq n \| x \|_\infty.$$

Proof.

Starting from the definition of $\nu_p$ we have

$$\| x \|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \leq n^\frac{1}{p} \max_{1 \leq i \leq n} |x_i| = n^\frac{1}{p} \| x \|_\infty.$$

The first inequality is immediate.
Corollary

Let $p$ and $q$ be two numbers such that $p, q \geq 1$. There exist two constants $c, d \in \mathbb{R}_{>0}$ such that

$$c \| \mathbf{x} \|_q \leq \| \mathbf{x} \|_p \leq d \| \mathbf{x} \|_q$$

for $\mathbf{x} \in \mathbb{R}^n$.

Proof.

Since $\| \mathbf{x} \|_\infty \leq \| \mathbf{x} \|_p$ and $\| \mathbf{x} \|_q \leq n \| \mathbf{x} \|_\infty$, it follows that $\| \mathbf{x} \|_q \leq n \| \mathbf{x} \|_p$. Exchanging the roles of $p$ and $q$, we have $\| \mathbf{x} \|_p \leq n \| \mathbf{x} \|_q$, so

$$\frac{1}{n} \| \mathbf{x} \|_q \leq \| \mathbf{x} \|_p \leq n \| \mathbf{x} \|_q$$

for every $\mathbf{x} \in \mathbb{R}^n$. \qed
Corollary

For every \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) and \( p \geq 1 \), we have \( d_\infty(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq nd_\infty(\mathbf{x}, \mathbf{y}) \). Further, for \( p, q > 1 \), there exist \( c, d \in \mathbb{R}_{>0} \) such that

\[
    cd_q(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq cd_q(\mathbf{x}, \mathbf{y})
\]

for \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \).
If $p \leq q$, then the closed sphere $B_{d_p}(x, r)$ is included in the closed sphere $B_{d_q}(x, r)$. For example, we have

$$B_{d_1}(0, 1) \subseteq B_{d_2}(0, 1) \subseteq B_{d_\infty}(0, 1).$$

In (a) - (c) we represent the closed spheres $B_{d_1}(0, 1)$, $B_{d_2}(0, 1)$, and $B_{d_\infty}(0, 1)$.
Theorem

Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ be $2m$ nonnegative numbers such that
\[ \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i = 1 \] and let $p$ and $q$ be two positive numbers such that
\[ \frac{1}{p} + \frac{1}{q} = 1. \]
We have
\[
\sum_{j=1}^{m} x_j^{\frac{1}{p}} y_j^{\frac{1}{q}} \leq 1.
\]

Proof.

The Hölder inequality applied to $x_1^{\frac{1}{p}}, \ldots, x_m^{\frac{1}{p}}$ and $y_1^{\frac{1}{q}}, \ldots, y_m^{\frac{1}{q}}$ yields the needed inequality
\[
\sum_{j=1}^{m} x_j^{\frac{1}{p}} y_j^{\frac{1}{q}} \leq \sum_{j=1}^{m} x_j \sum_{j=1}^{m} y_j = 1.
\]