Finite Automata and Regular Languages
(part VI)

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For any regular language $L$ there are several automata that are capable of recognizing it. Naturally, we are interested in finding among these automata the ones that have the smallest number of states.
Definition

The *Nerode equivalence* of a language \( L \subseteq A^* \) is the relation:

\[
\nu_L = \{(x, y) \in A^* \times A^* \mid xw \in L \text{ if and only if } yw \in L \text{ for every } w \in A^*\}.
\]
The relation $\nu_L$ is a right-invariant equivalence relation. In other words: if $(x, y) \in \nu_L$, then $(xu, yu) \in \nu_L$ for every $u \in A^*$. In terms of equivalence classes, $[x]_{\nu_L} = [y]_{\nu_L}$ implies $[xu]_{\nu_L} = [yu]_{\nu_L}$ for every $u \in A^*$.

Recall that $[x]_{\nu_L}$ denotes the $\nu_L$-equivalence class of the word $x$. The set of all equivalence classes of $\nu_L$ will be denoted by $A^*/\nu_L$. 
Lemma

Let $L \subseteq A^*$ be a language over an alphabet $A$. We have $(x, y) \in \nu_L$ if and only if $x^{-1}L = y^{-1}L$. 
Let $x, y \in A^*$ such that $(x, y) \in \nu_L$, and let $t \in x^{-1}L$. This means that $xt \in L$, which implies $yt \in L$ because of the definition of $\nu_L$. Therefore, $t \in y^{-1}L$, so $x^{-1}L \subseteq y^{-1}L$. The reverse inclusion can be obtained in the same manner, so $x^{-1}L = y^{-1}L$.

Conversely, if $x^{-1}L = y^{-1}L$, then $xt \in L$ if and only if $yt \in L$ for every $t \in A^*$, which means that $(x, y) \in \nu_L$. 
Definition

Let \( L \subseteq A^* \) be a language over the alphabet \( A \). The set of left derivatives of \( L \) is the set \( Q_L = \{ t^{-1}L \mid t \in A^* \} \).
Lemma

Let $L \subseteq A^*$ be a language over an alphabet $A$. The set of left derivatives of $L$ is finite if and only if $A^*/\nu_L$ is finite.

Proof.

The function $h_L : A^*/\nu_L \rightarrow Q_L$ defined by $h_L([x]_{\nu_L}) = x^{-1}L$ is a bijection. The desired conclusion follows immediately.
Lemma

Any language $L \subseteq A^*$ is a $\nu_L$-saturated set.

Proof.

In order to prove that $L$ is $\nu_L$-saturated, it suffices to show that the $\nu_L$-equivalence class of every $x \in L$ is included in $L$. Let $x \in L$. If $(x, y) \in \nu_L$, then $yz \in L$ whenever $xz \in L$ for any $z \in A^*$. Selecting $z = \lambda$ gives the required result.
Note that the previous lemma is equivalent to saying that for all words $x, y \in A^*$, if $x \in L$ and $x^{-1}L = y^{-1}L$, then $y \in L$.

No assumption is made about the language $L$; in particular, $L$ need not be regular.
Definition

Let $L \subseteq A^*$ be a language over the alphabet $A$. The automaton of the language $L$ is the deterministic automaton $M_L = (A, Q_L, \delta_L, L, F_L)$ is defined by $\delta_L(t^{-1}L, a) = (ta)^{-1}L$ for $t \in A^*$ and $a \in A$, and $F_L = \{x^{-1}L \mid x \in L\}$. 
The mapping $\delta_L$ is well defined; that is, $t^{-1}L = y^{-1}L$ implies $(ta)^{-1}L = (ya)^{-1}L$. Indeed, let $w \in (ta)^{-1}L$. We have $taw \in L$ which implies $aw \in t^{-1}L = y^{-1}L$. Consequently, $yaw \in L$, so $w \in (ya)^{-1}L$. Thus, $(ta)^{-1}L \subseteq (ya)^{-1}L$. The reverse inclusion can be shown similarly, so $(ta)^{-1}L = (ya)^{-1}L$.

We have $\delta(t^{-1}L, a) = a^{-1}(t^{-1}L)$ for every $t \in A^*$ and $a \in A$. This remark is very important for the algorithm discussed next.
We have
\[ \delta^*_L(x^{-1}L, y) = (xy)^{-1}L \]
for every \( x, y \in A^* \).
The argument is by induction on \( \ell = |y| \). The basis case, \( \ell = 0 \), is immediate. Suppose that the equality holds for words of length less than \( \ell \), and let \( y \) be a word of length \( \ell \). We have \( y = za \), where \( z \in A^* \), \( a \in A \) and \( |z| = \ell - 1 \). This gives:

\[
\begin{align*}
\delta^*_L(x^{-1}L, y) &= \delta^*_L(x^{-1}L, za) = \delta_L(\delta^*_L(x^{-1}L, z), a) \\
&= \delta_L((xz)^{-1}L, a) = (xza)^{-1}L = (xy)^{-1}L
\end{align*}
\]
The set of final states of $\mathcal{M}_L$ can now be written as

$$F_L = \{ \delta^*(L, x) \mid x \in L \},$$

which allows us to compute the set $F_L$, once we have computed the transition function.
Nerode’s Theorem:

Theorem

The language $L$ is regular if and only if the set $Q_L$ is finite.
Proof

Suppose that the set $Q_L$ is finite. In this case $M_L$ is a dfa and we have

$$L(M_L) = \{ x \in A^* \mid \delta^*_L(L, x) \in F_L \} = \{ x \in A^* \mid x^{-1}L \in F_L \}.$$

From the definition of $F_L$ it follows that $x \in L(M_L)$ implies that $x^{-1}L = z^{-1}L$ for some word $z \in L$, which shows that $(x, z) \in \nu_L$. Since $L$ is a $\nu_L$-saturated set, this implies $x \in L$. The reverse inclusion, $L \subseteq L(M_L)$ is immediate, and it is left to the reader. Therefore, $L$ is accepted by the dfa $M_L$, so $L$ is regular.

Conversely, suppose that $L$ is a regular language. The finiteness of the set $Q_L$ follows from a previous Corollary.
Theorem

Let $L$ be a regular language. The automaton $M_L$ has the least number of states among all dfas that accept $L$. 
Proof

Let $M = (A, Q, \delta, q_0, F)$ be a dfa such that $L = L(M)$. We intend to show that $|Q_L| \leq |Q|$. Clearly, if $M$ is to be minimal, it must be accessible, otherwise the automaton resulting from removing inaccessible states accepts the same language but has fewer states. In other words, we assume that for every state $q \in Q$ there exists a word $t \in A^*$ such that $\delta^*(q_0, t) = q$.

Define the mapping $f : Q \longrightarrow Q_L$ by $f(q) = t^{-1}L$ if $\delta^*(q_0, t) = q$. 

Proof (cont’d)

We need to verify that $f$ is well-defined, that is, that $\delta^*(q_0, u) = \delta^*(q_0, v)$ implies $u^{-1}L = v^{-1}L$. If $x \in u^{-1}L$, then $ux \in L$, that is, $\delta^*(q_0, ux) \in F$. Since $\delta^*(q_0, ux) = \delta^*(\delta^*(q_0, u), x)$ and $\delta^*(q_0, u) = \delta^*(q_0, v)$, it follows that $\delta^*(\delta^*(q_0, v), x) = \delta^*(q_0, vx) \in F$, so $vx \in L$ and $x \in v^{-1}L$. The reverse implication can be obtained by exchanging $u$ and $v$, so $f$ is indeed well-defined.

It is clear that the mapping $f$ is surjective, so $|Q_L| \leq |Q|$, which shows that $M_L$ has the least number of states among all dfas that accept the language $L$. 
The Algorithm

**Input:** A regular language $L$ over an alphabet $A$.

**Output:** The set $Q_L$ of left derivatives of $L$.

**Method:** Construct an increasing chain $Q_0, \ldots, Q_k, \ldots$ of finite subsets of $Q_L$ as follows:

$$Q_0 = \{ L \}$$

$$Q_{k+1} = Q_k \cup \{ a^{-1}K \mid a \in A \text{ and } K \in Q_k \}$$

Continue until $Q_{k+1} = Q_k$; then stop and output $Q_k$. 
Proof of Correctness:
The algorithm must stop, since \( Q_L \) is a finite set. It is easy to see that 
\( K \in Q_p \) if and only if the set \( K \) (considered as a state of the automaton 
\( M_L \)) can be reached by a word of length less than or equal to \( p \) in \( M_L \). Every state of the automaton \( M_L \) can be reached through a word of length 
less than \( |Q_L| \). Therefore, when the algorithm stops, all members of \( Q_L \) have been computed.
We recall several equalities previously shown that are useful in the computation of left derivatives of languages. Namely, if $L, K$ are two languages over the alphabet $A$ and $a \in A$, then we have:

\[
\begin{align*}
    a^{-1}(L \cup K) &= a^{-1}L \cup a^{-1}K \\
    a^{-1}(LK) &= (a^{-1}L)K \cup (L \cap \{\lambda\})a^{-1}K \\
    a^{-1}L^* &= (a^{-1}L)L^*
\end{align*}
\]
Example

Let \( A = \{a, b\} \). Consider the regular language \( L \) that consists of all words from \( A^* \) that contain the infix \( aba \). In other words, \( L = A^*abaA^* \).
We have $Q_0 = \{L\}$ and $Q_1 = Q_0 \cup \{a^{-1}L, b^{-1}L\}$. Note that

$$a^{-1}L = a^{-1}(A^*abaA^*)$$
$$= (a^{-1}A^*)(abaA^*) \cup (A^* \cap \{\lambda\})a^{-1}(abaA^*)$$
$$= A^*abaA^* \cup baA^*$$
$$= L \cup baA^*$$

and

$$b^{-1}L = b^{-1}(A^*abaA^*)$$
$$= (b^{-1}A^*)(abaA^*) \cup (A^* \cap \{\lambda\})b^{-1}(abaA^*)$$
$$= A^*abaA^*$$
$$= L,$$

because $b^{-1}(abaA^*) = \emptyset$, $(A^* \cap \{\lambda\})b^{-1}(abaA^*) = \emptyset$, and $b^{-1}A^* = A^*$. 
Thus,

\[ Q_1 = \{ L, L \cup baA^* \} \).

Next, in order to compute \( Q_2 \), observe that

\[ a^{-1} baA^* = \emptyset \]
\[ b^{-1} baA^* = aA^* \].

We obtain:

\[ a^{-1}(L \cup baA^*) = a^{-1}L = L \cup baA^* \]
\[ b^{-1}(L \cup baA^*) = b^{-1}L \cup aA^* = L \cup aA^* \],
To compute $Q_2$, observe that

\[
\begin{align*}
a^{-1} baA^* &= \emptyset \\
b^{-1} baA^* &= aA^*.
\end{align*}
\]

We obtain:

\[
\begin{align*}
a^{-1}(L \cup baA^*) &= a^{-1}L = L \cup baA^* \\
b^{-1}(L \cup baA^*) &= b^{-1}L \cup aA^* = L \cup aA^*.
\end{align*}
\]

The collection $Q_2$ is

\[
Q_2 = \{L, L \cup baA^*, L \cup aA^*\}
\]
Now we have

\[ a^{-1}aA^* = A^* \]
\[ b^{-1}aA^* = \emptyset, \]

which allows us to write:

\[ a^{-1}(L \cup aA^*) = a^{-1}L \cup A^* = A^* \]
\[ b^{-1}(L \cup aA^*) = b^{-1}L = L. \]

The collection \( Q_3 \) is given by

\[ Q_3 = \{ L, L \cup baA^*, L \cup aA^*, A^* \}. \]
Since $a^{-1}A^* = b^{-1}A^* = A^*$, it follows that $\mathcal{Q}_4 = \mathcal{Q}_3$, so

$$\mathcal{Q}_L = \{L, L \cup baA^*, L \cup aA^*, A^*\}.$$ 

The automaton $\mathcal{M}_L$ is defined by the following table:

<table>
<thead>
<tr>
<th>Input</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L$</td>
</tr>
<tr>
<td>$a$</td>
<td>$L \cup baA^*$</td>
</tr>
<tr>
<td>$b$</td>
<td>$L$</td>
</tr>
</tbody>
</table>
Since $F_L = \{ \delta^*(L, x) \mid x \in L \}$, we can compute $F_L$ by determining those members of $Q_L$ that can be reached from the initial state $L$ using words from $L$ of length not greater than 3.

The language $L$ contains only one word of length 3, namely $aba$, so $F_L = \{ \delta^*(L, aba) \} = \{ A^* \}$. The graph of $M_L$ is given next.