Context-Free languages
(part III)

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UMB
1. Type 3 Grammars and Finite Automata

2. The Case of One-Symbol Alphabet

3. Other Closure Properties of $\mathcal{L}_2$
The main result of this section is a proof that the class $\mathcal{R}$ of regular languages coincides with $\mathcal{L}_3$.

**Theorem**

Let $G$ be a type-3 grammar, and let $L$ be the language generated by $G$. There is a transition system $\mathcal{T}$ such that $L = L(\mathcal{T})$. 
Proof

Suppose that $G = (A_N, A_T, S, P)$ is a type-3 grammar. Define the transition system $\mathcal{T} = (A_T, A_N \cup \{Z\}, \theta, S, \{Z\})$, where $Z$ is a new symbol, $Z \notin A_N \cup A_T$, and

\[
\theta = \{(X, u, Y) \mid X \rightarrow uY \in P\} \\
\cup \{(X, u, Z) \mid X \rightarrow u \in P\}.
\]
Let $w \in L(G)$. There exists a derivation

$$
S \xrightarrow{G} u_0X_{i_0} \xrightarrow{G} u_0u_1X_{i_1} \cdots \xrightarrow{G} u_0u_1 \cdots u_{n-1}X_{i_{n-1}} \xrightarrow{G} u_0u_1 \cdots u_{n-1}u_n,
$$

where $w = u_0 \cdots u_{n-1}u_n$. The productions used in this derivation are $S \rightarrow u_0X_{i_0}$, $X_{i_{p-1}} \rightarrow u_pX_{i_p}$ for $1 \leq p \leq n-1$, and $X_{i_{n-1}} \rightarrow u_n$. Therefore, the triples

$$(S, u_0, X_{i_0}), (X_{i_0}, u_1, X_{i_1}), \ldots, (X_{i_{n-2}}, u_{n-1}, X_{i_{n-1}}), (X_{i_{n-1}}, u_n, Z)$$

must all be in $\theta$, which implies that $(S, u_0 \cdots u_n, Z) \in \theta^*$. Since $Z$ is a final state of $\mathcal{T}$, we have $u \in L(\mathcal{T})$, so $L(G) \subseteq L(\mathcal{T})$. 
(Proof cont’d)

Conversely, if \( u \in L(\mathcal{T}) \), then \((S, u, Z) \in \theta^*\). Taking into account the definition of \( \theta \), there are \( n \) intermediate states in \( \mathcal{T}, X_{i_0}, \ldots, X_{i_{n-1}} \) such that \( u = u_0 \cdots u_n \) and the triples

\[
(S, u_0, X_{i_0}), (X_{i_0}, u_1, X_{i_1}), \ldots, (X_{i_{n-2}}, u_{n-1}, X_{i_{n-1}}), (X_{i_{n-1}}, u_n, Z)
\]

exist in \( \theta \). This implies the existence in \( P \) of the productions

\[
S \to u_0X_{i_0}, X_{i_0} \to u_1X_{i_1}, \ldots, X_{i_{n-2}} \to u_{n-1}X_{i_{n-1}}, X_{i_{n-1}} \to u_n
\]

Using these productions we obtain the derivation

\[
S \stackrel{G}{\Rightarrow} u_0X_{i_0} \quad \Rightarrow \quad u_0u_1X_{i_1} \quad \cdots \quad \Rightarrow \quad u_0u_1 \cdots u_{n-1}X_{i_{n-1}} \Rightarrow G
\]

which implies that \( x \in L(\mathcal{T}) \). This proves the inclusion \( L(\mathcal{T}) \subseteq L(G) \).
Theorem

For every regular language $L$ there is a type-3 grammar $G$ such that $L(G) = L$.

Proof.

Let $\mathcal{M} = (A, Q, \delta, q_0, F)$ be a dfa such that $L = L(\mathcal{M})$. The type-3 grammar $G = (Q, A, q_0, P)$ whose productions are

\begin{align*}
q &\to aq' \quad \text{for each } q, q', a \text{ with } q' = \delta(q, a) \\
q &\to \lambda \quad \text{for each } q \in F.
\end{align*}

generates $L(\mathcal{M})$. 

Corollary

The class $\mathcal{L}_3$ coincides with the class $\mathcal{R}$ of regular languages.
Recall the Pumping Lemma for context-free languages:

**Theorem**

Let $L$ be a context-free language. There exists a number $n_L \in \mathbb{N}$ such that if $w \in L$ and $|w| \geq n_L$, then we can write

$$w = xyzu$$

such that $|y| \geq 1$ or $|u| \geq 1$, $|yzu| \leq n_L$ and $xy^nzu^n \in L$ for all $n \in \mathbb{N}$.

This is a necessary condition for the “context-freeness” of a language.
Let $A = \{a\}$ be an one-symbol alphabet.

- Word concatenation in $A^*$ is commutative.
- The formulation of the Pumping Lemma in this special case:
  Let $L$ be a context-free language. There exists a number $n_L \in \mathbb{N}$ such that if $w \in L$ and $|w| \geq n_L$, then we can write

$$w = rs$$

such that $1 \leq |s| \leq n_G$ and $rs^n \in L(G)$ for all $n \in \mathbb{N}$.

Note that $r \in L$ (since we can take $n = 0$).
If $|r| > n_L$ the same pumping lemma can be applied to $r$, and $r = r_1w_1$ with $|w_1| \leq n_L$ such $r_1w_1^{n_1} \in L$ for $n_1 \in \mathbb{N}$. Again $r_1 \in L$ (for $n = 0$), etc. This leads to a stronger form of the Pumping Lemma for languages over one-symbol alphabets.

If $L$ is a context-free language on an one-symbol alphabet, there exists a number $n_L$ such that every word $w \in L$ with $|x| \geq n_L$ can be written as

$$w = rs_1s_2 \cdots s_k,$$

where $|r|, |s_1|, \ldots, |s_k| \leq n_L$ and

$$rs_1^{n_1} \cdots s_k^{n_k} \in L$$

for $n_1, \ldots, n_k \in \mathbb{N}$.
Note that the set $K_n(L)$ of words in $L$ shorter than $n_L$ is finite, so it is regular. Since $L = (L \cap K_n(L)) \cup (L - K_n(L))$. The set $L - K_n(L)$ has the form $\{w_1, w_2, \ldots, w_n\}^*$, where $w_1, \ldots, w_n$ are the words that can be “pumped”. Thus, $L$ is a regular language.
Theorem

Let \( s : A^* \rightarrow B^* \) be a substitution. If \( s(a) \) is a context-free language for every \( a \in A \) and \( L \subseteq A^* \) is a context-free language, then \( s(L) \) is a context-free language.
Proof

Suppose that $L = L(G)$, where $G = (A_N, A, S, P)$ is a context-free grammar and let $s(a)$ is generated by the context-free grammar $G_a = (A^a_N, B, S_a, P_a)$ for $a \in A$.

We may assume that the sets of nonterminal symbols $A^a_N$ are pairwise disjoint.

Let $P'$ be the set of productions obtained from $P$ as follows. In each production of $P$ replace every letter $a \in A$ by the nonterminal $S_a$. We claim that the language $s(L)$ is generated by the grammar $G' = (A_N \cup \bigcup_{a \in A} A^a_N, B, S, P' \cup \bigcup_{a \in A} P_a)$. 
(Proof cont’d)

Let $y \in s(L)$. There exists a word $x = a_{i_0} \ldots a_{i_{n-1}} \in L$ such that $y \in s(x)$. This means that $y = y_0 \ldots y_{n-1}$, where $y_k \in s(a_{i_k}) = L(G_{a_{i_k}})$ for $0 \leq k \leq n - 1$. Thus, we have the derivations $S_{a_{i_k}} \xrightarrow{G_{a_{i_k}}} y_k$ for $0 \leq k \leq n - 1$, and the same derivations can be done in $G'$. Consequently, we obtain the derivation

$$S \xrightarrow{G'} S_{a_{i_0}} \ldots S_{a_{i_{n-1}}} \xrightarrow{G'} y_0 \ldots y_{n-1} = y,$$

which implies $y \in L(G')$, so $s(L) \subseteq L(G')$. 
Conversely, if \( y \in L(G') \), then any derivation \( S \xrightarrow{* \ G'} y \) is of the previous form.

The word \( y \) can be written as \( y = y_0 \cdots y_{n-1} \), where \( S_{a_{i_k}} \xrightarrow{* \ G'} y_k \) for \( 0 \leq k \leq n - 1 \), so \( y_k \in L(G_{a_{i_k}}) = s(a_{i_k}) \) for \( 0 \leq k \leq n - 1 \). This implies \( y = y_0 \cdots y_{n-1} \in s(a_{i_0} \cdots s(a_{i_{n-1}}) = s(x) \in s(L) \), so \( L(G') \subseteq s(L) \).

Since \( s(L) = L(G') \), it follows that \( s(L) \) is a context-free language.
Corollary

If $h : A^* \rightarrow B^*$ is a morphism and $L \subseteq A^*$ is a context-free language, then $h(L)$ is a context-free language.
The class $\mathcal{L}_2$ is closed with respect to inverse morphic images. In other words, if $h : B^* \longrightarrow A^*$ is a morphism, and $L \subseteq A^*$ is a context-free language, then $h^{-1}(L)$ is a context-free language.
Proof

Suppose that $B = \{ b_0, \ldots, b_{m-1} \}$ and that $h(b_i) = x_i$ for $0 \leq i \leq m - 1$. Let $B' = \{ b'_0, \ldots, b'_{m-1} \}$, and let $s$ be the substitution given by $s(a) = B'^* a B'^*$ for $a \in A$. 

\[
\begin{align*}
B &= \{ b_0, \ldots, b_{m-1} \} \\
B^* &\rightarrow A^* \\
s(a) &= B'^* a B'^* \\
B' &= \{ b'_0, \ldots, b'_{m-1} \}
\end{align*}
\]
Consider the finite language $H = \{b_ix_i | 0 \leq i \leq m\}$ in $(B' \cup A)^*$ and the mapping $g : \mathcal{P}(A^*) \longrightarrow \mathcal{P}((A \cup B')^*)$ given by $g(L) = s(L) \cap H^*$.

Define $h_1 : (A \cup B')^* \longrightarrow (\{c\} \cup B)^*$ and $h_2 : (\{c\} \cup B)^* \longrightarrow B^*$ by $h_1(a) = c$ for $a \in A$, $h_1(b') = b$ for all $b' \in B'$, and $h_2(c) = \lambda$, $h_2(b) = b$ for $b \in B$. 

\[ B = \{b_0, \ldots, b_{m-1}\} \]
\[ B' = \{b'_0, \ldots, b'_{m-1}\} \]
\[ H = \{b_ix_i | 0 \leq i \leq m\} \]
We claim that for every language $L \in \mathcal{P}(A)$ such that $\lambda \not\in L$, $h^{-1}(L) = h_2(h_1(g(L)))$ and hence, $h^{-1}(L)$ is context-free. This follows from the following equivalent statements:

1. $u = b_{i_0} \cdots b_{i_{k-1}} \in h^{-1}(L)$;
2. $h(u) = x_{i_0} \cdots x_{i_{k-1}} \in L$;
3. $b_{i_0}^\prime x_{i_0} \cdots b_{i_{k-1}}^\prime x_{i_{k-1}} \in g(L)$;
4. $h_1(b_{i_0}^\prime x_{i_0} \cdots b_{i_{k-1}}^\prime x_{i_{k-1}}) = b_{i_0} c \cdots c \cdots b_{i_{k-1}} c \cdots c \in h_1(g(L))$;
5. $h_2(b_{i_0} c \cdots c \cdots b_{i_{k-1}} c \cdots c) = b_{i_0} \cdots b_{i_{k-1}} = u \in h_2(h_1(g(L)))$. 
If $\lambda \in L$, the language $L - \{\lambda\}$ is context-free, so $h^{-1}(L - \{\lambda\})$ is also context-free. Note that $h^{-1}(L) = h^{-1}(L - \{\lambda\}) \cup h^{-1}(\{\lambda\})$ and that $h^{-1}(\{\lambda\}) = \{a \in A \mid h(a) = \lambda\}^*$. Since $h^{-1}(\{\lambda\})$ is regular it follows that $h^{-1}(L)$ is context-free.
Reminder

We defined the shuffle of languages

**Definition**

Let $A$ be an alphabet and let $G, K$ be two languages over $A$. The *shuffle* of $G$ and $K$ is the language

$$\text{shuffle}(G, K) = \{x_0y_0x_1y_1\cdots x_{n-1}y_{n-1} \mid x_0x_1\cdots x_{n-1} \in G \text{ and } y_0y_1\cdots y_{n-1} \in K\}.$$
We proved

**Theorem**

*There is an alphabet $B$ and there exist three morphisms $g, k, h$ from $B^*$ to $A^*$ such that $h$ is a very fine morphism, $g, k$ are fine morphisms and $\text{shuffle}(G, K) = h(g^{-1}(G) \cap k^{-1}(K))$.***
Corollary

Let \( L \subseteq A^* \) be a context-free language and let \( R \subseteq A^* \) be a regular language. Then, \( \text{shuffle}(L, R) \) is a context-free language.