1 Partitions and Equivalence Relations

2 Partitions
Definition
An *equivalence relation* on a set $S$ is a relation $\rho$ that is reflexive, symmetric, and transitive.
This means that

- $(x, x) \in \rho \text{ for every } x \in S$;
- $(x, y) \in \rho \text{ if and only if } (y, x) \in \rho$;
- $(x, y) \in \rho \text{ and } (y, z) \in \rho \text{ imply } (x, z) \in \rho$. 

Example

Let $U$ and $V$ be two sets, and consider a function $f : U \to V$. The relation $\ker(f) \subseteq U \times U$, called the *kernel* of $f$, is given by

$$\ker(f) = \{(u, u') \in U \times U \mid f(u) = f(u')\}.$$ 

In other words, $(u, u') \in \ker(f)$ if $f$ maps both $u$ and $u'$ into the same element of $V$. 
Example

Let $m \in \mathbb{N}$ be a positive natural number. Define the function $f_m : \mathbb{Z} \longrightarrow \mathbb{N}$ by $f_m(n) = r$ if $r$ is the remainder of the division of $n$ by $m$. The range of the function $f_m$ is the set $\{0, \ldots, m - 1\}$.

The relation $\ker(f_m)$ is usually denoted by $\equiv_m$. We have $(p, q) \in \equiv_m$ if and only if $p - q$ is divisible by $m$; if $(p, q) \in \equiv_m$, we also write $p \equiv q(\text{mod } m)$. 
Definition

Let \( \rho \) be an equivalence on a set \( U \) and let \( u \in U \). The \textit{equivalence class} of \( u \) is the set \([u]_\rho\), given by

\[
[u]_\rho = \{ y \in U \mid (u, y) \in \rho \}.
\]

When there is no risk of confusion, we write simply \([u]\) instead of \([u]_\rho\).
Note that an equivalence class \([u]\) of an element \(u\) is never empty since \(u \in [u]\) because of the reflexivity of \(\rho\).

**Theorem**

*Let \(\rho\) be an equivalence on a set \(U\) and let \(u, v \in U\). The following three statements are equivalent:*

1. \((u, v) \in \rho\);
2. \([u] = [v]\);
3. \([u] \cap [v] \neq \emptyset\).
**Definition**

Let $S$ be a set and let $\rho \in \text{EQ}(S)$. A subset $U$ of $S$ is $\rho$-saturated if it equals a union of equivalence classes of $\rho$.

It is easy to see that $U$ is a $\rho$-saturated set if and only if $x \in U$ and $(x, y) \in \rho$ imply $y \in U$. It is clear that both $\emptyset$ and $S$ are $\rho$-saturated sets.
Definition

Let $S$ be a nonempty set. A \textit{partition} of $S$ is a nonempty collection $\pi = \{B_i \mid i \in I\}$ of nonempty subsets of $S$, such that $\bigcup\{B_i \mid i \in I\} = S$, and $B_i \cap B_j = \emptyset$ for every $i, j \in I$ such that $i \neq j$.

Each set $B_i$ of $\pi$ is a block of the partition $\pi$.

The set of partitions of a set $S$ is denoted by $\text{PART}(S)$. The partition of $S$ that consists of all singletons of the form $\{s\}$ with $s \in S$ will be denoted by $\alpha_S$; the partition that consists of the set $S$ itself will be denoted by $\omega_S$. 
Example

For the two-element set $S = \{a, b\}$, there are two partitions: the partition $\alpha_S = \{\{a\}, \{b\}\}$ and the partition $\omega_S = \{\{a, b\}\}$.

For the one-element set $T = \{c\}$, there exists only one partition, $\alpha_T = \omega_T = \{\{t\}\}$.
Example

A complete list of partitions of a set $S = \{a, b, c\}$ consists of

\[
\pi_0 = \{\{a\}, \{b\}, \{c\}\}, \quad \pi_1 = \{\{a, b\}, \{c\}\}, \\
\pi_2 = \{\{a\}, \{b, c\}\}, \quad \pi_3 = \{\{a, c\}, \{b\}\}, \\
\pi_4 = \{\{a, b, c\}\}.
\]

Clearly, $\pi_0 = \alpha_S$ and $\pi_4 = \omega_S$. 
Definition

Let $S$ be a set and let $\pi, \sigma \in \text{PART}(S)$. The partition $\pi$ is \textit{finer} than the partition $\sigma$ if every block $C$ of $\sigma$ is a union of blocks of $\pi$. This is denoted by $\pi \preceq \sigma$. 
Theorem

Let \( \pi = \{ B_i \mid i \in I \} \) and \( \sigma = \{ C_j \mid j \in J \} \) be two partitions of a set \( S \). For \( \pi, \sigma \in \text{PART}(S) \), we have \( \pi \leq \sigma \) if and only if for every block \( B_i \in \pi \) there exists a block \( C_j \in \sigma \) such that \( B_i \subseteq C_j \).
Proof

If $\pi \subseteq \sigma$, then it is clear for every block $B_i \in \pi$ there exists a block $C_j \in \sigma$ such that $B_i \subseteq C_j$.

Conversely, suppose that for every block $B_i \in \pi$ there exists a block $C_j \in \sigma$ such that $B_i \subseteq C_j$. Since two distinct blocks of $\sigma$ are disjoint, it follows that for any block $B_i$ of $\pi$, the block $C_j$ of $\sigma$ that contains $B_i$ is unique.

Therefore, if a block $B$ of $\pi$ intersects a block $C$ of $\sigma$, then $B \subseteq C$.

Let $Q = \bigcup\{B_i \in \pi \mid B_i \subseteq C_j\}$. Clearly, $Q \subseteq C_j$. Suppose that there exists $x \in C_j - Q$. Then, there is a block $B_\ell \in \pi$ such that $x \in B_\ell \cap C_j$, which implies that $B_\ell \subseteq C_j$. This means that $x \in B_\ell \subseteq C$, which contradicts the assumption we made about $x$. Consequently, $C_j = Q$, which concludes the argument.
Note that $\alpha_S \leq \pi \leq \omega_S$ for every $\pi \in \text{PART}(S)$.

Two equivalence classes either coincide or are disjoint. Therefore, starting from an equivalence $\rho \in \text{EQ}(U)$, we can build a partition of the set $U$.

**Definition**

The *quotient set* of the set $U$ with respect to the equivalence $\rho$ is the partition $U/\rho$, where

$$U/\rho = \{[u]_\rho \mid u \in U\}.$$

An alternative notation for the partition $U/\rho$ is $\pi_\rho$. 
Theorem

Let \( \pi = \{B_i \mid i \in I\} \) be a partition of the set \( U \). Define the relation \( \rho_\pi \) by \((x, y) \in \rho_\pi\) if there is a set \( B_i \in \pi \) such that \( \{x, y\} \subseteq B_i \). The relation \( \rho_\pi \) is an equivalence.
Proof

Let $B_i$ be the block of the partition that contains $u$. Since $\{u\} \subseteq B_i$, we have $(u, u) \in \rho_\pi$ for any $u \in U$, which shows that $\rho_\pi$ is reflexive. The relation $\rho_\pi$ is clearly symmetric. To prove the transitivity of $\rho_\pi$, consider $(u, v), (v, w) \in \rho_\pi$. We have the blocks $B_i$ and $B_j$ such that $\{u, v\} \subseteq B_i$ and $\{v, w\} \subseteq B_j$. Since $v \in B_i \cap B_j$, we obtain $B_i = B_j$ by the definition of partitions; hence, $(u, w) \in \rho_\pi$. 